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MODERN GEOMETRY

OF THE

POINT, STRAIGHT LINE, AND CIRCLE

MODERN GEOMETRY

OF THE

POINT, STRAIGHT LINE, AND CIRCLE

AN ELEMENTARY TREATISE

BY

J. A. THIRD, M.A.

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WILLIAM BLACKWOOD AND SONS
EDINBURGH AND LONDON
MDCCCXCVIII

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It is unnecessary to indicate the numerous original sources of information drawn upon, as the material used may now be regarded as common property; but special indebtedness has to be acknowledged to Townsend's 'Modern Geometry,' and to the 'Syllabus of Modern Plane Geometry' issued by the Association for the Improvement of Geometrical Teaching.

The reader is strongly recommended to draw the necessary figures for himself, according to the directions given in the text. Figures gradually produced by the student himself are more readily grasped than those presented to him in a complete form, and tend to facilitate the understanding and remembering of the connected propositions. With this in view, several of the articles have been left without diagrams.

A collection of exercises is given at the end. These have been arranged roughly in the order in which the subjects are discussed in the text.

J. A. T.

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MODERN GEOMETRY.

CHAPTER I.

LINE SEGMENTS. CONVENTION OF POSITIVE AND NEGATIVE. PRINCIPLE OF GENERALITY.

(1) *Def.*—If A and B are two points on a straight line, the part of the straight line intercepted between A and B is called a *line segment*, or briefly a *segment*.

If the segment is regarded as measured from A to B it is denoted by AB; if from B to A, by BA.

A segment is regarded as either *positive* (+) or *negative* (−), according to the direction in which it is measured. If segments measured in one direction be regarded as positive, segments measured in the opposite direction are regarded as negative. Thus if AB is taken as positive, BA is negative; if BA is taken as positive, AB is negative. In either case we have the following relations:—

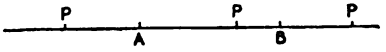
$$AB = -BA; \quad -AB = BA; \quad AB + BA = 0.$$

Consistently with the conception of a segment as a positive or negative quantity, the laws and operations of Algebra are regarded as applicable to segments. In some cases the resulting expressions are capable of geometrical interpretation, in other cases, not. *E.g.*, AB, BC, CD, DE being segments of a straight line, AB^2 can be interpreted as the square on AB, and AB.CD as the rectangle whose sides are

equal to AB and CD, while the expressions AB.BC.CD.DE and AB/BC.CD, taken by themselves, have no geometrical meaning. It will be noticed that AB^2 is always positive, while AB.CD and AB/CD are positive or negative according as AB and CD have the same or different signs.

(2) THEOREM.—*If* A *and* B *are two points on a straight line and* P *any other point on the line, then*

$$AP+PB=AB.$$



Proof.—

(i) If P lies between A and B (in which case AB is said to be *internally* divided in P), the statement is obviously true.

(ii) If P lies to the left of A (in which case AB is said to be *externally* divided in P), then

$$PB=PA+AB.$$
$$\therefore\ AP+PB=AP+PA+AB.$$
But $$AP+PA=0. \qquad (Art.\ 1.)$$
$$\therefore\ AP+PB=AB.$$

(iii) If P lies to the right of A (in which case also AB is said to be *externally* divided in P), then

$$AP=AB+BP.$$
$$\therefore\ AP+PB=AB+BP+PB.$$
But $$BP+PB=0. \qquad (Art.\ 1.)$$
$$\therefore\ AP+PB=AB.$$

Cor.— $$AP-BP=AB.$$

Proof.— $$-BP=PB. \qquad (Art.\ 1.)$$
$$\therefore\ AP-BP=AP+PB=AB.$$

(3) Let A, B (fig. of *Art.* 2) be two fixed points on a straight line, taken as points of reference, and P a point free to move along the line. Consider the variations in magnitude and sign of the ratio AP/PB as P traverses the line.

When P is at A (which may be taken as the starting-point of its journey), AP/PB = 0.

As P moves from A to B, AP/PB is *positive* and continuously *increases* in magnitude. When P bisects AB, AP/PB = +1.

When P is at B, AP/PB = ∞.

At the moment when P passes from B towards the right, AP/PB becomes *negative*, and as P moves on towards the right, AP/PB remains *negative* and continuously *diminishes* in magnitude.

When P reaches infinity to the right, AP/PB = −1. This may be proved as follows:—

$$\frac{AP}{PB} = \frac{AB - PB}{PB} = \frac{AB}{PB} - 1. \qquad (\textit{Art. 2, Cor.})$$

But PB = ∞, and AB is a finite quantity.

$$\therefore\ AB/PB = 0.$$

$$\therefore\ AP/PB = -1.$$

When P is at infinity to the left, AP/PB still = −1. The proof is exactly the same as that just given.

As P moves from infinity on the left towards A, AP/PB remains *negative* and continuously *diminishes* in magnitude, until, when P arrives again at A, it = 0.

(4) It thus appears that as P moves from A to B, the ratio AP/PB assumes successively all *positive* values between 0 and ∞, and that as P moves over the rest of the line, from B to infinity on the right, and from infinity on the left back to A, the ratio AP/PB assumes successively all *negative* values between ∞ and 0. It will also be observed that for no two positions of P on the line is the ratio AP/PB the same in both magnitude and sign. Hence for any given value of the ratio AP/PB, there is only *one* position of P. In order to make this statement generally true, and to reconcile it with the fact that when AP/PB = −1, P may be either at infinity on the right or at infinity on the left, we regard the point at an infinite distance beyond B as identically the same as the point at an infinite

distance beyond A—*i.e.*, we regard *a straight line as having one and only one point at infinity.* The line AB may therefore be regarded as traced out by a point P which moves continuously, without break, from A towards the right, through the point at infinity, back again to A. We thus arrive at the notion of a straight line as a *continuous locus.*

Def.—The ratio AP/PB which determines the position of P on the line, with reference to A and B, is sometimes called the *position ratio* of P.

(5) The two points A and B on the line divide it into *two* segments, one internal and finite, the other external, infinite, and containing the point at infinity. Each of these segments is the complement of the other.

Since $AP/PB = -1$ when P is the point at infinity, the point at infinity bisects the external segment, and may be said to bisect AB *externally.*

(6) If A, B, C, D are four points taken in order on a straight line, A and C are separated from each other by B, and B and D are separated from each other by C. A and B, B and C, C and D are obviously not separated from each other. The same is true of D and A, for these points are connected by the unbroken segment containing the point at infinity. Thus of four points on a line, only two pairs of points are separated from each other.

The following are a few theorems on the relations existing among the segments of a straight line. Others will be given in the exercises.

(7) If A, B, C are three points on a straight line, occurring in any order, then

$$AB + BC + CA = 0.$$

Proof.—$AB + BC = AC$, whatever be the relative positions of A, B, C. (*Art.* 2.)

$\therefore AB + BC + CA = AC + CA = 0.$ (*Art.* 1.)

Cor.—If A, B, C, D are four points on a straight line, occurring in any order, then

$$AB + BC + CD + DA = 0.$$

Proof.—
$$\begin{aligned} &AB+BC+CD+DA \\ &=AC+CD+DA && (\textit{Art. } 2.) \\ &=AD+DA && (\textit{Art. } 2.) \\ &=0. && (\textit{Art. } 1.) \end{aligned}$$

The proposition may be extended to any number of points in a line, the letters in the statement being always taken in cyclic order. The proof consists in repeated applications of the principle proved in *Art.* 2.

(8) If A, B, C are any three points on a straight line, and M the middle point of AB, then
$$MC=\tfrac{1}{2}(AC+BC).$$

Proof.—
$$\begin{aligned} &AC=AM+MC. \\ &BC=BM+MC. && (\textit{Art. } 2.) \\ &\therefore AC+BC=AM+BM+2MC. \\ &\text{But } AM=MB. \quad \therefore AM+BM=0. \\ &\therefore AC+BC=2MC. \end{aligned}$$

(9) If A, B, C, D are any four points on a straight line, M the middle point of AB, and N the middle point of CD, then
$$MN=\tfrac{1}{2}(AC+BD)=\tfrac{1}{2}(AD+BC).$$

Proof.—
$$\begin{aligned} &MB+BC+CN+NM=0. && (\textit{Art. } 7.) \\ &\therefore MN=MB+BC+CN \\ &\quad=\tfrac{1}{2}(AB+BC+BC+CD) \\ &\quad=\tfrac{1}{2}(AC+BD) && (\textit{Art. } 2.) \\ &\quad=\tfrac{1}{2}(AD+BC). && (\textit{Art. } 7.) \end{aligned}$$

(10) If A, B, C, D are four points on a straight line, occurring in any order, then
$$AB.CD+BC.AD+CA.BD=0.$$

Proof.—
$$\begin{aligned} &AB+BC+CD+DA=0. && (\textit{Art. } 7.) \\ &\therefore AD=AB+BC+CD. \\ &\therefore AB.CD+BC.AD \\ &=AB.CD+BC(AB+BC+CD) \\ &=AB.CD+BC.AB+BC^2+BC.CD \\ &=AB(BC+CD)+BC(BC+CD) \\ &=(AB+BC)(BC+CD) \\ &=AC.BD=-CA.BD. \\ &\therefore AB.CD+BC.AD+CA.BD=0. \end{aligned}$$

(11) If A, B, C, D are any four points on a straight line, then

$$AB.CD^2+BC.AD^2+CA.BD^2=-AB.BC.CA.$$

Proof.— $CD=-BC+BD.$ (*Art.* 2.)

$\therefore CD^2=BC^2+BD^2-2BC.BD.$

$\therefore AB.CD^2=AB.BC^2+AB.BD^2-2AB.BC.BD.$

$AD=AB+BD.$ (*Art.* 2.)

$\therefore AD^2=AB^2+BD^2+2AB.BD.$

$\therefore BC.AD^2=BC.AB^2+BC.BD^2+2AB.BC.BD.$

$\therefore$ by addition,

$$AB.CD^2+BC.AD^2=AB.BC(AB+BC)+(AB+BC)BD^2.$$
$$=-AB.BC.CA-CA.BD^2.$$

Dividing by AB.BC.CA, we obtain another form of the theorem :—

$$\frac{CD^2}{BC.CA}+\frac{AD^2}{CA.AB}+\frac{BD^2}{AB.BC}=-1.$$

(12) The use of the signs positive and negative may be extended to segments which do not lie in the same straight line. Thus in the figure of *Art.* 183, AB, BC, CA are regarded as having relatively to each other the same signs as their respectively corresponding segments A′B′, B′C′, C′A′.

The sides of similar rectilineal figures may be regarded as having the same or different signs, according as they are, or are not, measured the same way round.

(13) It will be noticed that the adoption of the convention of positive and negative as applied to segments enables us to make geometrical statements more general and comprehensive than would otherwise be the case. For example, the theorem of *Art.* 2, which, apart from the rule of signs, would not be true in cases ii. and iii., is by the employment of the convention rendered true for all cases. This greater comprehensiveness of statement, by which as many cases as possible are included in a general enunciation, is one of the chief distinguishing features of modern as contrasted with ancient geometry.

The desire for generality has given rise to many other

modifications and extensions of Euclid's conceptions, among which the following may be noted here. Two parallel straight lines are regarded as meeting at a point at infinity. A tangent to a circle is regarded as meeting the circle in two coincident points. Two circles in contact are regarded as meeting each other in two coincident points. Adopting these conceptions, we are in a position to make the following statements, which would otherwise not be generally true: (i) *Every* two straight lines in a plane meet at a point. (ii) *Every* straight line in the same plane with a circle, and meeting that circle at a point, meets it at another point. (iii) *Every* circle in the same plane with another circle, and meeting that circle at a point, meets it at another point. The last two statements may be made even yet more general by the introduction, frequent in modern geometry, of the idea of *imaginary* elements. Thus every straight line in the same plane with a circle either meets the circle in two points or does not meet it at all, but if in the latter case we conceive the line as meeting the circle in two *imaginary*, as distinguished from *real* or actual points, we have the following general theorem: Every straight line in the same plane with a circle meets that circle in two points (which are both real and distinct, or both real and coincident, or both imaginary). Similarly every circle in the same plane with another circle meets that circle in two points (which are both real and distinct, or both real and coincident, or both imaginary).

It is sometimes useful in the interests of generality to regard the three fundamental figures of elementary geometry — the point, the straight line, and the circle — as being particular cases of the same figure. This is done by conceiving a point as a circle whose radius is infinitely small and a straight line as a circle whose radius is infinitely great.

The mode in which these principles are applied will be illustrated in subsequent chapters.

CHAPTER II.

COLLINEARITY AND CONCURRENCY. POINTS, STRAIGHT LINES, AND CIRCLES CONNECTED WITH THE TRIANGLE.

(14) *Defs.*—

A straight line drawn across a system of lines is called a *transversal.*

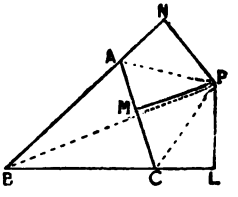
Three or more straight lines which intersect in the same point are said to be *concurrent.*

Three or more points lying on the same straight line are said to be *collinear.*

Four or more points lying on the circumference of the same circle are said to be *concyclic.*

Three straight lines of unlimited length (*sides*) and their three points of intersection (*vertices*) form a *triangle.*

(15) THEOREM.—*If* L, M, N *are three points on the sides* BC, CA, AB *respectively of a triangle* ABC, *such that the*



perpendiculars to the sides at these points are concurrent, then $(BL^2 - LC^2) + (CM^2 - MA^2) + (AN^2 - NB^2) = 0.$

Proof.—Join P, the point of concurrence, with A, B, C.

Then $(BL^2-LC^2)+(CM^2-MA^2)+(AN^2-NB^2)$

$=BL^2+PL^2-(PL^2+LC^2)+CM^2+PM^2-(PM^2+MA^2)$

$+AN^2+PN^2-(PN^2+NB^2)$

$=BP^2-CP^2+CP^2-AP^2+AP^2-BP^2=0.$

Converse.—If L, M, N are points on the sides BC, CA, AB respectively, of a triangle ABC, such that

$$(BL^2-LC^2)+(CM^2-MA^2)+(AN^2-NB^2)=0,$$

then the perpendiculars to the sides at L, M, N are concurrent.

Proof.—Let the perpendiculars to BC and CA intersect in P, and let N′ be the foot of the perpendicular from P on AB.

Then, by the above theorem,

$$(BL^2-LC^2)+(CM^2-MA^2)+(AN'^2-N'B^2)=0.$$

But $(BL^2-LC^2)+(CM^2-MA^2)+(AN^2-NB^2)=0.$

$\therefore AN'^2-N'B^2=AN^2-NB^2.$

$\therefore$ N′ is the same point as N.

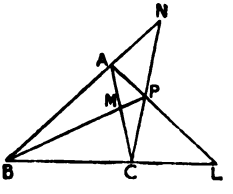
This converse supplies us with an important criterion of concurrency.

Cor.—The three perpendiculars at the middle points A′, B′, C′ of the sides BC, CA, AB respectively, of a triangle ABC, are concurrent.

The point of concurrence, O, is the centre of the circle circumscribed to the triangle, briefly called the *circumcircle* (*Euc.* iv. 5), and is called the *circumcentre.*

(16) THEOREM (Ceva's).—*If straight lines drawn through the vertices,* A, B, C, *of a triangle are concurrent, and meet the opposite sides in* L, M, N *respectively, then*

$$\frac{BL}{LC}\cdot\frac{CM}{MA}\cdot\frac{AN}{NB}=1.$$



Proof.—Let P be the point of concurrence.

$$\triangle APB:\triangle BPL=AP:PL=\triangle CPA:\triangle CPL.$$

$$\therefore \triangle APB:\triangle CPA=\triangle BPL:\triangle CPL=BL:LC.$$

Similarly, $\triangle BPC : \triangle APB = CM : MA$,
and $\triangle CPA : \triangle BPC = AN : NB$.

$$\therefore \frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = \frac{\triangle APB}{\triangle CPA} \cdot \frac{\triangle BPC}{\triangle APB} \cdot \frac{\triangle CPA}{\triangle BPC} = 1.$$

This result is often written for convenience:

$$BL.CM.AN = LC.MA.NB.$$

This mode of expression must not be taken as implying that any geometrical meaning is to be assigned to such products.

The segments BL, LC, &c., are so measured that the product of the three ratios BL/LC, &c., may be positive. If CL were substituted for LC and the other segments left unchanged, the theorem would be untrue, the signs of segments being taken into account. It will be observed that in the figure given, BL/LC and AN/NB are negative, while CM/MA is positive. If the figure be drawn so that P shall lie within the triangle, all the ratios will be positive.

Converse.—If L, M, N are points on the sides BC, CA, AB respectively, of a triangle ABC, such that

$$\frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = 1,$$

then AL, BM, CN are concurrent.

Proof.—Let P be the point of intersection of AL and BM, and through P draw CP to cut AB in N′.

Then, by the above theorem,

$$\frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN'}{N'B} = 1.$$

But $$\frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = 1.$$

$\therefore AN'/N'B = AN/NB.$

$\therefore$ N′ is the same point as N. (*Art.* 4.)

This converse supplies us with another valuable criterion of concurrency.

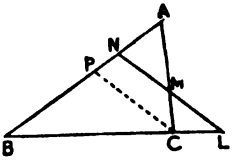
Cor.—The straight lines joining the vertices A, B, C

of a triangle, to the middle points A′, B′, C′ respectively of the opposite sides, meet in a point G.

Def.—AA′, BB′, CC′ are called the *medians*, and the point G the *centroid* of the triangle.

(17) THEOREM (Menelaus').—*If a transversal meet the sides* BC, CA, AB *of a triangle* ABC, *in* L, M, N *respectively, then*

$$\frac{BL}{LC}\cdot\frac{CM}{MA}\cdot\frac{AN}{NB}=-1.$$



Proof.—Through one of the vertices, say C, draw a straight line parallel to the transversal, and meeting the opposite side, say AB in P.

Then BL/LC = BN/NP *; CM/MA = PN/NA.

$$\therefore \frac{BL}{LC}\cdot\frac{CM}{MA}\cdot\frac{AN}{NB}=\frac{BN}{NP}\cdot\frac{PN}{NA}\cdot\frac{AN}{NB}.$$

$$=\frac{BN}{NB}\cdot\frac{PN}{NP}\cdot\frac{AN}{NA}=(-1)^3=-1.$$

This result is often written

$$BL.CM.AN=-LC.MA.NB.$$

Converse.—If L, M, N are points on the sides BC, CA, AB respectively, of a triangle ABC, such that

$$\frac{BL}{LC}\cdot\frac{CM}{MA}\cdot\frac{AN}{NB}=-1,$$

then L, M, N are collinear.

Proof.—Let the straight line joining L and M meet AB in N′.

Then, by the above theorem,

$$\frac{BL}{LC}\cdot\frac{CM}{MA}\cdot\frac{AN'}{N'B}=-1.$$

$$\text{But } \frac{BL}{LC}\cdot\frac{CM}{MA}\cdot\frac{AN}{NB}=-1.$$

* BL/LC = BN/PN would be incorrect, one ratio being positive, the other negative.

$\therefore$ AN′/N′B = AN/NB.
$\therefore$ N′ is the same point as N. (*Art* 4.)

This converse supplies us with a criterion of collinearity.

(18) THEOREM.—*The three points at infinity on any three non-parallel straight lines in a plane are collinear.*

Proof.—Let the three lines form a triangle ABC, and let L, M, N be the points at infinity on the sides BC, CA, AB respectively.

Then BL/LC = −1, CM/MA = −1, AN/NB = −1. (*Art.* 3.)

$$\therefore \frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = (-1)^3 = -1.$$

$\therefore$ L, M, N are collinear. (*Art* 17.)

Cor.—Since *any three* points at infinity in a plane are collinear, it follows that *all the points* at infinity in a plane are collinear.

Def.—The straight line on which all the points at infinity in a plane lie is called the *straight line at infinity.* Just as a straight line has only one point at infinity, so a plane has only one straight line at infinity.

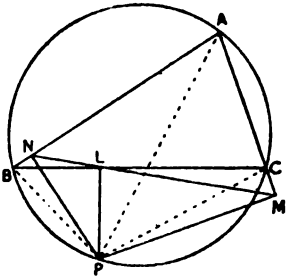
The corresponding theorem, that every three, and therefore all, parallel straight lines meet in a point at infinity, may be similarly proved by an application of Ceva's theorem. The theorem obviously follows also from the conception in modern geometry that two parallel straight lines meet in a point at infinity, combined with the fact (*Art.* 4) that each straight line has only one point at infinity.

(19) THEOREM.—*The feet of the perpendiculars drawn to the sides of a triangle from any point on the circumcircle are collinear.*

Proof.—Let P be any point on the circumcircle of the triangle ABC, and let L, M, N be the feet of the perpendiculars from P to BC, CA, AB respectively. Join PA, PB, PC.

Then since A, B, C, P are concyclic, angle PCM = angle PBN; angle PBL = angle PAM; angle PAN = angle PCL.

And the angles at L, M, N are right.



$\therefore$ the following are pairs of similar triangles:—

PCM, PBN; PBL, PAM; PAN, PCL.

$$\frac{BL}{LC}\cdot\frac{CM}{MA}\cdot\frac{AN}{NB}=\frac{BL}{MA}\cdot\frac{CM}{NB}\cdot\frac{AN}{LC}$$

$$=(\text{by similar triangles})\ \frac{LP}{PM}\cdot\frac{MP}{PN}\cdot\frac{NP}{PL}\ *$$

$$=\frac{LP}{PL}\cdot\frac{MP}{PM}\cdot\frac{NP}{PN}=(-1)^3=-1.$$

$\therefore$ L, M, N are collinear.† (*Art.* 17.)

Def.—The transversal LMN is called the *pedal line* of the point P with respect to the triangle ABC.

(20) THEOREM.—*The three perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.*

* See second paragraph of *Art.* 12.

† This theorem, here given as an application of Menelaus' theorem, may be more shortly proved thus: P, L, C, M; P, B, A, C; P, B, N, L are quartets of concyclic points. $\therefore$ angle PLM = angle PCM = angle PBN = supplement of angle PLN. $\therefore$ angle PLN + angle PLM = 2 right angles.

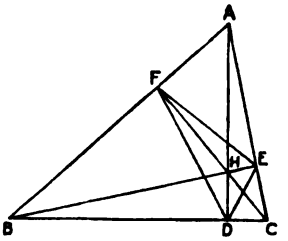
Proof.—Let D, E, F be the feet of the perpendiculars drawn from the vertices A, B, C respectively of a triangle to the opposite sides.

Then $(BD^2-DC^2)+(CE^2-EA^2)+(AF^2-FB^2)$
$=AB^2-CA^2+BC^2-AB^2+CA^2-BC^2=0.$

$\therefore$ AD, BE, CF are concurrent. (*Art.* 15.)

Defs.—AD, BE, CF are called the *altitudes* of the triangle.

H, the point of concurrence, is called the *orthocentre.*



The triangle DEF joining the feet of the perpendiculars is called the *orthocentric* or *pedal triangle.*

The following are important theorems, the proofs of which the student will have no difficulty in supplying for himself:—

(ii) A, B, C are the orthocentres of the triangles BHC, CHA, AHB respectively.

(iii) The triangles ABC, BHC, CHA, AHB have the same pedal triangle, DEF.

(iv) The following are quartets of concyclic points: B, C, E, F; C, A, F, D; A, B, D, E; B, D, H, F; C, E, H, D; A, F, H, E.

(v) The following are quartets of similar triangles: ABC, AEF, DBF, DEC; BCE, CAD, BHD, AHE; CAF, ABE, CHE, BHF; BCF, ABD, CHD, AHF.

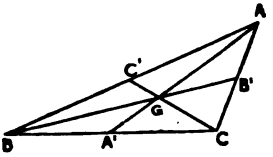
(vi) The altitudes of a triangle bisect the angles of its pedal triangle.

(vii) $AH.HD=BH.HE=CH.HF$; $BA.AF=HA.AD=CA.AE$; $CB.BD=HB.BE=AB.BF$; $AC.CE=HC.CF=BC.CD$.

(21) THEOREM.—*If* AA′, BB′, CC′ *are the medians of a triangle* ABC, *and* G *its centroid, then*

$$AG = 2GA', \quad BG = 2GB', \quad CG = 2GC'.$$

Proof.—The sides of the triangle ABA′ are cut by the transversal CGC′.

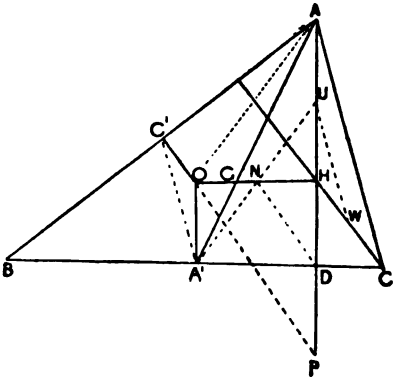


$$\therefore \text{ (Art. 17) } \frac{AG.A'C.BC'}{GA'.CB.C'A} = -1. \quad \therefore \frac{AG}{GA'} = \frac{BC.C'A}{A'C.BC'} = 2.$$

Similarly BG/GB′ = 2, and CG/GC′ = 2.

(22) THEOREM.—*If* H *be the orthocentre and* O *the circumcentre of a triangle* ABC, *and* A′, B′, C′, *the middle points of* BC, CA, AB, *respectively then*

$$AH = 2OA', \quad BH = 2OB', \quad CH = 2OC'.$$



Proof.—Join A′C′, and UW the middle points of AH and CH. Then A′C′ is parallel to CA and equal to $\frac{1}{2}$CA, and WU is parallel to CA and equal to $\frac{1}{2}$CA. ∴ A′C′ and WU are parallel and equal. Again, OA′ and UH are parallel, being both perpendicular to BC, and OC′ and

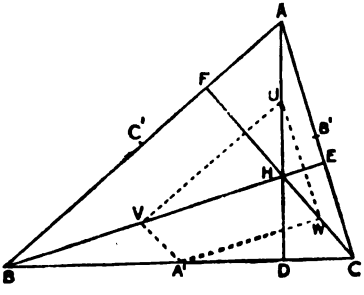
WH are parallel, being both perpendicular to AB. ∴ the triangles OA′C′ and HUW are equiangular; and they have a side of the one equal to a side of the other. ∴ they are congruent. ∴ OA′ = UH = ½AH, and OC′ = WH = ½CH. Similarly it may be proved that OB′ = ½BH.

(23) THEOREM.—*The orthocentre, centroid, and circumcentre of a triangle are collinear.*

Proof.—In the figure of the last article, join AA′ containing the centroid G, OG, GH. Then angle HAG = angle OA′G, and AG : GA′ = 2 : 1 = AH : OA′. ∴ triangles OA′G and HAG are equiangular (*Euc.* vi. 5). ∴ angle OGA′ = angle HGA. ∴ OG, GH are in the same straight line.

Cor.— HG = 2GO.

(24) THEOREM.—*If* AD, BE, CF *be the altitudes, and* H *the orthocentre of a triangle* ABC, *then the middle points of the sides of the triangles* ABC, BHC, CHA, AHB *lie on the circle circumscribed to their pedal triangle* DEF.



Proof.—Let A′, B′, C′, U, V, W be the middle points of BC, CA, AB, AH, BH, CH respectively. Join A′W, WU, UV and VA′. Then, since the line joining the mid-points of the sides of a triangle is parallel to the base, A′W, WU, UV, and VA′ are parallel to BH, CA, AB, and HC respectively. But BH is perpendicular to CA, and AB is perpendicular to HC. ∴ A′WU and UVA′ are right angles. And A′DU is a right angle. ∴ the circle having A′U as diameter passes through A′, D, W, U, V—*i.e.*, the circle described about UVW passes through A′, D. Similarly we can prove that it passes

through B′, E and C′, F. The circle thus passes through the *nine points*, D, E, F, A′, B′, C′, U, V, W.

Def. — Hence the circle circumscribed to the pedal triangle of a triangle is called the *nine-point circle* of the latter.

Cor. i.—A′U, B′V, C′W, are diameters of the nine-point circle, and are concurrent.

Cor. ii.—Triangles ABC, BHC, CHA, AHB have the same nine-point circle. (*Art.* 20, iii.)

(25) THEOREM.—*The orthocentre, circumcentre, and the centre of the nine-point circle of a triangle are collinear.*

Proof.—In the figure of *Art.* 22 join A′U meeting OH in N. Then, since OA′, AD are parallel, triangles OA′N and HUN are equiangular. But OA′ = UH. ∴ the triangles are congruent, and A′N = NU. But A′U is a diameter of the nine-point circle (*Art.* 24, i.) ∴ N is the centre. And H, O, N are collinear.

Cor. i.—The orthocentre, centroid, circumcentre, and the centre of the nine-point circle of a triangle are collinear. (*Art.* 23).

Cor. ii.— ON = NH.

Cor. iii.— OG = 2GN.

Cor. iv.— NH = 3GN.

(26) THEOREM.—*The radius of the circumcircle of a triangle is equal to twice the radius of its nine-point circle.*

Proof.—In the figure of *Art.* 22 join OA. Then OA = radius of the circumcircle of ABC, and NU = radius of the nine-point circle. But, since N, U bisect HO, HA respectively, NU = ½OA.

Cor. i.—The circumcircles of the triangles ABC, BHC, CHA, AHB (figure of *Art.* 24) are equal.

Proof.—The radius of the circumcircle of each of the triangles named = twice the radius of the nine-point circle of the triangle. But the triangles have the same nine-point circle (*Art* 24, *Cor.* ii.) Therefore their circumcircles are equal.

Cor. ii.—If HD (figure of *Art.* 22) be produced to P, so that HD = DP, P is a point on the circumcircle of ABC.

Proof.—Join ND and OP. Since N, D bisect HO, HP respectively, ND = $\frac{1}{2}$OP. But ND is a radius of the nine-point circle. ∴ OP = radius of the circumcircle. ∴ since O is the circumcentre, P is on the circumcircle.

Cor. iii.—If HA′ (figure of *Art.* 22) be produced to Q, so that HA′ = A′Q, Q is a point on the circumcircle of ABC.

(27) THEOREM.—*If* R *denote the radius of the circumcircle of the triangle* ABC, △ *the area of the triangle, and* a, b, c *the lengths of the sides* BC, CA, AB *respectively, then* $4R\triangle = abc$.

Proof.—See figure of *Art.* 20.

$2R.AD = AB.CA = bc$. (*Addendum C to Euc.* vi.)

∴ $2R.a.AD = abc$. But $a.AD = 2\triangle$.

∴ $4R\triangle = abc$.

Cor.—

$$\frac{BC.CA.AB}{ABC} = \frac{BC.CH.HB}{BHC} = \frac{CA.AH.HC}{CHA} = \frac{AB.BH.HA}{AHB},$$

where the denominators stand for the areas of the triangles named.

Proof.—Each of the above fractions = 4R, R being the same for the four triangles ABC, BHC, CHA, AHB.

(28) THEOREM (Desargues'). — *If two triangles be so situated that their corresponding vertices connect concurrently, then their corresponding sides intersect collinearly.*

Proof.—Let AA′, BB′, CC′, the connectors of corresponding vertices of the triangles ABC, A′B′C′ meet in P; and let the pairs of corresponding sides BC B′C′; CA, C′A′; AB, A′B′, meet in L, M, N respectively.

Then the triangles BCP, CAP, ABP are cut by the transversals LC′B′, MA′C′, NB′A′ respectively.

∴ $BL.CC'.PB' = -LC.C'P.B'B$;

$CM.AA'.PC' = -MA.A'P.C'C$;

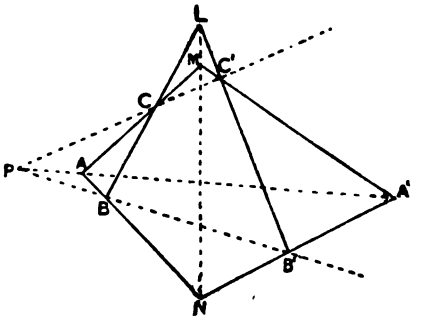
AN.BB′.PA′ = −NB.B′P.A′A. (*Art.* 17.)

∴ by multiplying and cancelling,

BL.CM.AN = −LC.MA.NB.

∴ L, M, N are collinear. (*Art.* 17, *converse.*)

Converse.—If two triangles be so situated that their corresponding sides intersect collinearly, then their corresponding vertices connect concurrently.



Proof.—In the above figure, let BB′, CC′ intersect in P. The vertices of the triangles BB′N, CC′M connect concurrently in L. ∴ their corresponding sides intersect collinearly—*i.e.*, A, A′, P are collinear. ∴ AA′ also passes through P.

Defs.—Two triangles placed as in the above theorem are said to be *in perspective.* The point of concurrence is called the *centre of perspective*, and the line of collinearity the *axis of perspective.*

(29) THEOREM.—*If three triangles are in perspective two and two, and if their three centres of perspective are coincident, then their three axes of perspective are concurrent.*

Proof.—Let ABC, A′B′C′, and A″B″C″ be the three triangles, and let the triads of points A, A′, A″; B, B′, B″; C, C′, C″ lie on three straight lines passing through a point O. Let P, Q, R be the points of intersection of the pairs AB, A′B′; A′B′, A″B″; A″B″, AB respectively, and P′, Q′,

R′ the points of intersection of the pairs BC, B′C′; B′C′, B″C″; B″C″ BC respectively.

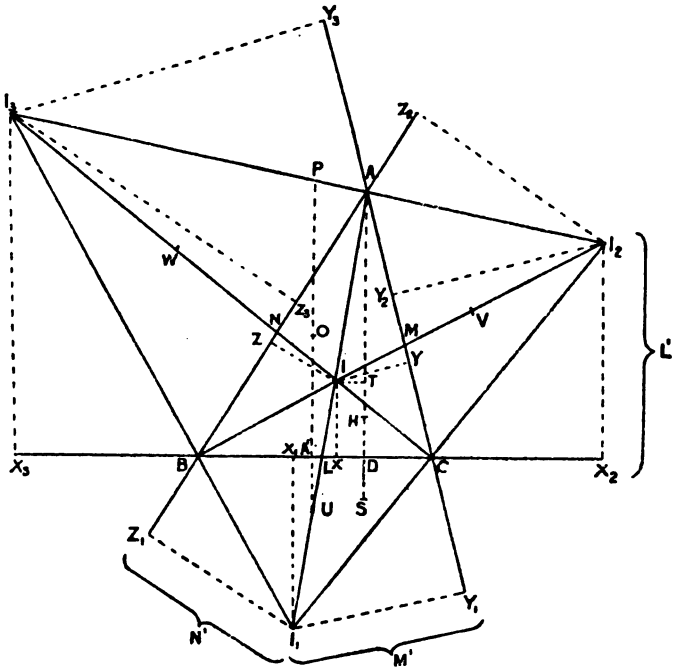
Then PP′, QQ′, RR′ are the three axes of perspective. The following pairs of sides of the triangles PQR, P′Q′R′ intersect collinearly in the points B, B′, B″ respectively—viz., RP, R′P′; PQ, P′Q′; QR, Q′R. ∴ PP′, QQ′, RR′ are concurrent. (*Art.* 28, *converse.*)

The correlative theorem, which may be proved similarly, is: If three triangles are in perspective two and two, and if their three axes of perspective are coincident then their three centres of perspective are collinear.

CHAPTER III.

POINTS, STRAIGHT LINES, AND CIRCLES CONNECTED WITH THE TRIANGLE—*continued.*

(30) *Defs.*—The straight line which passes through the



vertex of an angle, makes equal angles with its arms, and

lies within the angle, is called the *internal bisector* of the angle. The straight line which passes through the vertex of an angle, makes equal angles with its arms, and lies outside the angle, is called the *external bisector* of the angle. Thus in the subjoined figure AI is the internal bisector and AI_2 the external bisector of the angle BAC.

THEOREM.—*The three internal bisectors of the angles of a triangle are concurrent.*

Proof.—Let AL, BM, CN be the internal bisectors of the angles of the triangle ABC, L, M, N being the points where they meet the sides BC, CA, AB respectively.

Then, by *Euc.* vi. 3,

$$\frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = \frac{AB}{CA} \cdot \frac{BC}{AB} \cdot \frac{CA}{BC} = 1.$$

$\therefore$ AL, BM, CN are concurrent. (*Art.* 16.)

Let I denote the point of concurrence.

(31) THEOREM.—*The internal bisector of any angle of a triangle, and the external bisectors of the other two angles, are concurrent.*

Proof.—In the figure of *Art.* 30 let AL′, BM′, CN′ be the external bisectors of the angles A, B, C respectively, L′, M′, N′ being the points where they meet the sides BC, CA, AB respectively.

Then, by the extension of *Euc.* vi. 3,

$$\frac{BL}{LC} \cdot \frac{CM'}{M'A} \cdot \frac{AN'}{N'B} = \frac{AB}{CA} \cdot \frac{BC}{AB} \cdot \frac{CA}{BC} = 1.$$

$\therefore$ AL, BM′, CN′ are concurrent. (*Art.* 16.)

A similar proof holds for the other two cases. Let I_1, I_2, I_3 denote the respective points of concurrence of the following triads of lines:—

AL, BM′, CN′; BM, CN′, AL′; CN, AL′, BM′.

(32) THEOREMS.—*If from the points* I, I_1, I_2, I_3, *in the figure of* Art. 30, IX, I_1X_1, I_2X_2, I_3X_3 *be drawn perpendicular to* BC; IY, I_1Y_1, I_2Y_2, I_3Y_3 *perpendicular to* CA; IZ, I_1Z_1, I_2Z_2, I_3Z_3 *perpendicular to* AB, *then the following*

are triads of equal lines: IX, IY, IZ; I_1X_1, I_1Y_1, I_1Z_1; I_2X_2, I_2Y_2, I_2Z_2; I_3X_3, I_3Y_3, I_3Z_3.

These results can be easily proved by means of *Euc.* i. 26.

Cors.—

The circle with centre I and radius IX touches the sides of the triangle ABC at X, Y, Z.

The circle with centre I_1 and radius I_1X_1 touches the sides of the triangle ABC at X_1, Y_1, Z_1.

The circle with centre I_2 and radius I_2X_2 touches the sides of the triangle ABC at X_2, Y_2, Z_2.

The circle with centre I_3 and radius I_3X_3 touches the sides of the triangle ABC at X_3, Y_3, Z_3.

Defs.—Of these circles the first is called the *inscribed circle*, shortly the *incircle*, while the other three are called the *escribed circles*, shortly the *excircles*, of the triangle ABC. The centre of the incircle is called, for the sake of brevity, the *incentre*, while the centres of the excircles are called the *excentres*. r, r_1, r_2, r_3 denote the radii of the circles whose centres are I, I_1, I_2, I_3 respectively.

(33) In the following theorems referring to the figure of *Art.* 30, s denotes the semiperimeter of the triangle ABC, and a, b, c denote the lengths of the sides BC, CA, AB respectively. For the sake of symmetry in the lettering, the signs of segments have been disregarded, absolute lengths only being considered.

(i) $AY_1 = AZ_1 = BZ_2 = BX_2 = CX_3 = CY_3 = s$.

(ii) $AY = AZ = BZ_3 = BX_3 = CX_2 = CY_2 = s - a$;
$AY_3 = AZ_3 = BZ = BX = CX_1 = CY_1 = s - b$;
$AY_2 = AZ_2 = BZ_1 = BX_1 = CX = CY = s - c$.

(iii) $XX_3 = YY_3 = X_1X_2 = Y_1Y_2 = c$;
$YY_1 = ZZ_1 = Y_2Y_3 = Z_2Z_3 = a$;
$ZZ_2 = XX_2 = Z_1Z_3 = X_1X_3 = b$.

(iv) $XX_1 = b \sim c$; $YY_2 = c \sim a$; $ZZ_3 = a \sim b$.

(v) $X_2X_3 = b + c$; $Y_1Y_3 = c + a$; $Z_1Z_2 = a + b$.

(vi) $AY + AY_1 + AY_2 + AY_3 = AZ + AZ_1 + AZ_2 + AZ_3$
$= BZ + BZ_1 + BZ_2 + BZ_3 = BX + BX_1 + BX_2 + BX_3$
$= CX + CX_1 + CX_2 + CX_3 = CY + CY_1 + CY_2 + CY_3$
$= 2s$.

...ide the angle, is called the *internal bise...*
...angle. The straight line which passes through ...
...of an angle, makes equal angles with its arms, an...
...side the angle, is called the *external bisector* of t...
...Thus in the subjoined figure AI is the internal bise...
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THEOREM.—*The three internal bisectors of the ang...*
triangle are concurrent.

Proof.—Let AL, BM, CN be the internal bisecto...
the angles of the triangle ABC, L, M, N being the po...
where they meet the sides BC, CA, AB respectively.
Then, by *Euc.* vi. 3,

$$\frac{BL}{LC}\cdot\frac{CM}{MA}\cdot\frac{AN}{NB}=\frac{AB}{CA}\cdot\frac{BC}{AB}\cdot\frac{CA}{BC}=1.$$

∴ AL, BM, CN are concurrent.
Let I denote the point of concurrence.

(31) THEOREM.—*The internal bisector of* any a...
triangle, and the external bisectors of the ...
are concurrent.

Proof.—In the figure of *Art.* 30 let AL′, BM′, CN′
the external bisectors of the angles A, B, C respectively
L′, M′, N′ being the points where they meet the sides BC
CA, AB respectively.
Then, by the extension of *Euc.* vi. 3,

$$\frac{BL}{LC}\cdot\frac{CM'}{M'A}\cdot\frac{AN'}{N'B}=\frac{AB}{CA}\cdot\frac{BC}{BC}\cdot\frac{CA}{BC}=1.$$

∴ AL, BM′, CN′ are concurrent.
A similar proof holds for the ...
Let I_1, I_2, I_3 denote the ...
the following ...

are triads of equal lines: IX, IY, IZ; I_1X_1, I_1Y_1, I_1Z_1; I_2X_2, I_2Y_2, I_2Z_2; I_3X_3, I_3Y_3, I_3Z_3.

These results can be easily proved by means of *Euc.* i. 26.

Cors.—

The circle with centre I and radius IX touches the sides of the triangle ABC at X, Y, Z.

The circle with centre I_1 and radius I_1X_1 touches the sides of the triangle ABC at X_1, Y_1, Z_1.

The circle with centre I_2 and radius I_2X_2 touches the sides of the triangle ABC at X_2, Y_2, Z_2.

The circle with centre I_3 and radius I_3X_3 touches the sides of the triangle ABC at X_3, Y_3, Z_3.

Defs.—Of these circles the first is called the *inscribed circle*, shortly the *incircle*, while the other three are called the *escribed circles*, shortly the *excircles*, of the triangle ABC. The centre of the incircle is called, for the sake of brevity, the *incentre*, while the centres of the excircles are called the *excentres*. r, r_1, r_2, r_3 denote the radii of the circles whose centres are I, I_1, I_2, I_3 respectively.

(33) In the following theorems referring to the figure of *Art.* 30, s denotes the semiperimeter of the triangle ABC, and a, b, c denote the lengths of the sides BC, CA, AB respectively. For the sake of symmetry in the lettering, the signs of segments have been disregarded, absolute lengths only being considered.

(i) $AY_1 = AZ_1 = BZ_2 = BX_3 = CX_2 = CY_3 = s$.

(ii) $AY = AZ = BZ_2 = BX_3 = CX_2 = CY_3 = s-a$;

$AY_2 = AZ_3 = BZ = BX = CX_1 = CY_1 = s-$

$AY_3 = AZ_2 = BZ_1 = BX_1 = CX = CY = s$

(iii) XX

$Y_1Y_2 = Z_1Z_3 = a$;

$X_2X_3 = b$.

$ZZ_2 = a$

$Z_1Z_2 = a$;

$AZ + Z$

$BX = BX$

$CX_3 = C$

BX_2
$+CY_3$

Proofs.—

(i) $BZ_1 = BX_1$, each being a tangent from B to an excircle. Similarly $CY_1 = CX_1$. $\therefore AY_1 + AZ_1 = b + CX_1 + c + BX_1 = 2s$. $\therefore$, since AY_1, AZ_1 are tangents from A to an excircle, $AY_1 = AZ_1 = s$. Similarly the rest of the theorem may be proved.

(ii) $BZ = BX$, each being a tangent from B to the incircle. Similarly $CY = CX$. $\therefore AY + AZ = b - CX + c - BX = b + c - a = 2(s-a)$. $\therefore$, since AY, AZ are tangents from A to the incircle, $AY = AZ = s - a$. Similarly it may be proved that $BZ = BX = s - b$, and that $CX = CY = s - c$.

Again, $BX_3 = CX_3 - a = s - a$ (*Theorem* i). But BZ_3, BX_3 are tangents from B to an excircle. $\therefore BZ_3 = BX_3 = s - a$. Similarly the rest of the theorem may be proved.

(iii) $XX_3 = BX + BX_3 = s - b + s - a = c$ (*Th.* ii). Similarly the rest of the theorem may be proved.

(iv) $XX_1 = BX \sim BX_1 = (s-b) \sim (s-c) = b \sim c$ (*Th.* ii). Similarly the rest of the theorem may be proved.

(v) $X_2X_3 = BX_2 + BX_3 = s + s - a = b + c$ (*Ths.* i, ii). Similarly the rest of the theorem may be proved.

(vi) This is proved by substituting for the various segments their values in terms of s, a, b, c, as given in Theorems i and ii.

(34) THEOREM.—*The straight lines joining the vertices of a triangle to the points where the incircle, or any one of the excircles, touches the opposite sides, are concurrent.*

Proof.—In the figure of *Art.* 30,

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{s-b}{s-c} \cdot \frac{s-c}{s-a} \cdot \frac{s-a}{s-b} = 1. \quad (\textit{Art.}\ 33.)$$

$\therefore$ AX, BY, CZ are concurrent.

Similarly it may be proved that the following are triads of concurrent lines:—

AX_1, BY_1, CZ_1; AX_2, BY_2, CZ_2; AX_3, BY_3, CZ_3.

(35) THEOREM.—*Of the four points* I, I_1, I_2, I_3, *in the figure of* Art. 30, *any one is the orthocentre of the triangle formed by joining the other three.*

Proof.—Angle I_1AI_3 = angle I_1AB + angle I_3AB = angle I_1AC + angle I_2AC = angle I_1AI_2. ∴ angle I_1AI_3 = angle I_1AI_2 = a right angle. ∴ I_1I and I_2I_3 are perpendicular. Similarly it may be proved that the following are pairs of perpendicular lines :—

I_2I, I_3I_1, and I_3I and I_1I_2.

Compare this theorem with its converse. (*Art.* 20, vi.)

Cor. i.—ABC is the pedal triangle of the triangles $I_1I_2I_3$, II_2I_3, II_3I_1, II_1I_2.

Cor. ii.—The circumcircle of the triangle ABC is the nine-point circle of the triangles $I_1I_2I_3$, II_2I_3, II_3I_1, II_1I_2.

Cor. iii.—The circumcircle of the triangle ABC bisects the joins II_1, II_2, II_3, I_1I_2, I_2I_3, I_3I_1.

Cor. iv.—The following are sets of concyclic points: B, C, I, I_1; C, A, I, I_2; A, B, I, I_3.

(36) THEOREM.—

$$\triangle = rs = r_1(s-a) = r_2(s-b) = r_3(s-c),$$

where $\triangle$ *denotes the area of the triangle* ABC *in the figure of* Art. 30.

Compare *Art.* 27.

Proof.— $\triangle = BIC + CIA + AIB$

$= \frac{1}{2}ra + \frac{1}{2}rb + \frac{1}{2}rc = \frac{1}{2}r(a+b+c) = rs.$

Also, $\triangle = CI_1A + AI_1B - BI_1C$

$= \frac{1}{2}r_1b + \frac{1}{2}r_1c - \frac{1}{2}r_1a = \frac{1}{2}r_1(b+c-a) = r_1(s-a).$

Similarly the rest of the theorem may be proved.

Cor.— $$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}.$$

Proof.— $$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{s-a}{\triangle} + \frac{s-b}{\triangle} + \frac{s-c}{\triangle}$$

$$= \frac{3s-(a+b+c)}{\triangle} = \frac{3s-2s}{\triangle} = \frac{s}{\triangle} = \frac{1}{r}.$$

(37) THEOREM.—$4R = r_3 + r_2 + r_1 - r$, *where* R *denotes the radius of the circumcircle.*

Proof.—In the figure of *Art.* 30 let the circumcircle of triangle ABC bisect II_1 in U, and I_2 I_3 in P. Join UP.

Then UP is a diameter of the circumcircle (*Arts.* 24, i; 35, ii). Since angle BAU = angle UAC, arc BU = arc UC. ∴ diameter UP bisects BC at right angles, say in the point A′. ∴ UP is parallel to IX, I_1X_1, I_2X_2, and I_3X_3. ∴ triangles IXL, UA′L, I_1X_1L are equiangular; also triangles I_2X_2L', PA′L′, I_3X_3L' are equiangular.

∴ $UA'/I_1X_1 = UL/I_1L$; and $UA'/IX = UL/IL$.

∴ $UA'/(I_1X_1 - IX) = UL/(I_1L - IL) = UL/2UL = \frac{1}{2}$.

∴ $UA' = \frac{1}{2}(I_1X_1 - IX) = \frac{1}{2}(r_1 - r)$.

Also, $PA'/I_3X_3 = PL'/I_3L'$, and $PA'/I_2X_2 = PL'/I_2L'$.

∴ $PA'/(I_3X_3 + I_2X_2) = PL'/(I_3L' + I_2L') = PL'/2PL' = \frac{1}{2}$.

∴ $PA' = \frac{1}{2}(I_3X_3 + I_2X_2) = \frac{1}{2}(r_3 + r_2)$.

∴ $PA' + UA' = \frac{1}{2}(r_3 + r_2 + r_1 - r)$.

∴ $2R = \frac{1}{2}(r_3 + r_2 + r_1 - r)$.

In order to preserve symmetry in the lettering, the signs of linear segments have been disregarded in the above proof.

Cor. i.—If, in the figure of *Art.* 30, U, V, W be the points in which the circumcircle of the triangle ABC bisects II_1, II_2, II_3 respectively, then the perpendiculars from U, V, W on the sides BC, CA, AB respectively, bisect these sides.

Cor. ii.—If A′, B′, C′ be the middle points of the sides BC, CA, AB respectively, then

$$UA' + VB' + WC' = \tfrac{1}{2}(r_3 + r_2 + r_1 - 3r) = 2R - r.$$

Cor. iii.—If O be the circumcentre of the triangle ABC, then $OA' + OB' + OC' = R + r$.

Cor. iv.— $UB = UC = UI = UI_1$, &c.

The following four propositions deal with the distances between the orthocentre H, the circumcentre O, the in- and ex-centres I, I_1, &c., and the centre of the nine-point circle N of the triangle ABC.

(38) THEOREM.— $OH^2 = R^2 - 2AH.HD$.

Proof.—In the figure of *Art.* 30 draw AHD perpendicular to BC and produce it to meet the circumcircle in S. Then OS = OA. ∴ (*Euc.* iii. 35) OH^2
$= OA^2 - AH.HS = (Art.\ 26, ii)\ R^2 - 2AH.HD.$

(39) THEOREM.— $IO^2 = R^2 - 2Rr.$

Proof.—Using the figure of *Art.* 30 we have (*Euc.* iii. 35) $IO^2 = OA^2 - UI.IA = (Art.\ 37, iv.)\ OA^2 - UC.IA.$ But triangles AIY, PUC are similar. ∴ UC.IA = UP.IY.
∴ $IO^2 = OA^2 - UP.IY = R^2 - 2Rr.$ since UP is a diameter of the circumcircle.

Cor.— $I_1O^2 = R^2 + 2Rr_1$; $I_2O^2 = R^2 + 2Rr_2$;
$I_3O^2 = R^2 + 2Rr_3.$

(40) THEOREM.— $IH^2 = 2r^2 - AH.HD.$

Proof.—In the figure of *Art.* 30 draw IT perpendicular to AD.

Then angles BAI, I_1AC are equal, and (*Art.* 35, iv.) angles ABI, AI_1C are equal. ∴ triangles AIB, ACI_1 are similar. ∴ $AI.AI_1 = AB.AC = (Euc.$ vi., *Add. C.*$)2R.AD.$ That is $2AI.AU - AI^2 = 2R.AD$ (since $UI = UI_1$).

∴ $2AT.2R - AI^2 = 2R.AD$ (AIT, UAP are similar triangles).

That is $4R.AT = 2R.AD + AI^2.$

Again, $IH^2 = AI^2 + AH^2 - 2AH.AT$ (*Euc.* ii. 13).

$\therefore 2R.IH^2 = 2R.AI^2 + 2R.AH^2 - AH.4R.AT$
$= 2R.AI^2 + 2R.AH^2 - AH(2R.AD + AI^2)$
$= (2R - AH)AI^2 - 2R.AH(AD - AH)$
$= (2OU - 2OA')AI^2 - 2R.AH.HD$ (*Art.* 22.)
$= 2A'U.AI^2 - 2R.AH.HD$
$= 2r.UB.AI - 2R.AH.HD$ (BA'U, AYI are similar triangles)
$= 2r.r.2R - 2R.AH.HD$ (PBU, AYI are similar triangles).
$\therefore IH^2 = 2r^2 - AH.HD.$

Cor.— $I_1H^2 = 2r_1^2 - AH.HD;$
$I_2H^2 = 2r_2^2 - AH.HD;$
$I_3H^2 = 2r_3^2 - AH.HD.$

(iv) All parallels to MN (or BC) are antiparallel to BC (or MN) with respect to A;

(v) A straight line drawn through the point A to bisect MN (or BC) bisects every other antiparallel to BC (or MN).

The following two particular cases of antiparallels are important—

(vi) The sides of a triangle and the sides of its pedal triangle are antiparallel with respect to the angles of the original triangle (*Art.* 20, v.); and, conversely, if the sides of a triangle and the sides of an inscribed triangle are antiparallel with respect to the angles of the first, the inscribed triangle is the pedal triangle of the other.

Cor.—Antiparallels to the sides of a triangle are parallel to the sides of its pedal triangle, and conversely.

(vii) The tangent at A to the circumcircle of the triangle ABC is antiparallel to BC; and, conversely, the line drawn through A antiparallel to BC touches the circumcircle at A.

In this case the circle described about BCMN becomes the circumcircle of ABC, the chord MN becomes the tangent at A, and angle AMN becomes the angle (equal to angle ABC) which AC makes with the tangent at A.

Cor. a.—Antiparallels to BC are parallel to the tangent at A to the circumcircle of ABC, and conversely.

Cor. b.—Antiparallels to BC are perpendicular to the radius from A of the circumcircle of ABC, and conversely.

The following two theorems, the second of which may be deduced from the first, are also among the obvious properties of antiparallels to the sides of a triangle:—

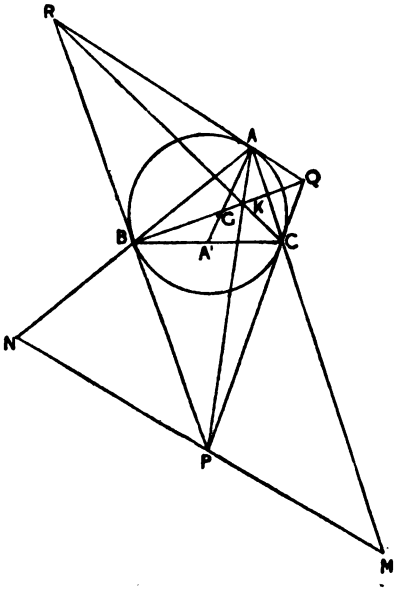
(viii) Antiparallels to two sides of a triangle meet the third side at equal angles.

(ix) If two antiparallels to two sides of a triangle bisect each other, they are equal.

(44) *Def.*—Straight lines drawn from the vertices of a

triangle so as to bisect the antiparallels (with respect to the angles at these vertices) of the opposite sides (*vide Art.* 43, v.) are called the *symmedians* of the triangle.

THEOREM.—*The symmedians of a triangle are concurrent.*



Proof.—Let ABC be the triangle, and let the tangents to the circumcircle of ABC, at the vertices, form the triangle PQR, P, Q, R being opposite to A, B, C respectively. Join AP, BQ, CR. Through P draw a line parallel to QR meeting CA in M and AB in N.

Then MN is antiparallel to BC (*Art.* 43, vii. *Cor. a*). ∴ angle CMP = angle ABC = angle QCA (*Euc.* iii. 32) = angle MCP. ∴ PM = PC. Similarly PN = PB. But PC = PB. ∴ AP bisects MN. ∴ AP is a symmedian (*Art.* 43, v.) Similarly BQ and CR are symmedians. But AP, BQ, CR connect the vertices of triangle PQR with

the points of contact of its incircle. ∴ (*Art.* 34) they are concurrent.

Def.—The point of concurrence is called the *symmedian point* (denoted by K) of the triangle ABC.

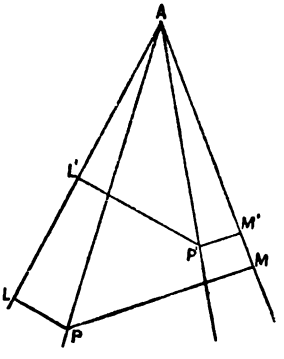
Cor.—Antiparallels through K to each side of a triangle are bisected at the point, and are equal (*Art.* 43, ix.)

(45) *Def.*—Two straight lines drawn through the vertex of an angle are said to be *isogonal conjugates* with respect to the angle when they are equally inclined to its bisector.

Obviously the angle between one of the conjugates and one of the arms of the angle of reference equals the angle between the other conjugate and the other arm of the angle of reference.

Isogonal conjugates are a particular case of antiparallels.

THEOREM.—*If a point be taken on each of two isogonal conjugates with respect to a given angle, and if perpendiculars be drawn from these two points to the two arms of the angle of reference, the rectangle contained by the perpendiculars to one arm is equal to the rectangle contained by the perpendiculars to the other arm.*



Proof.—Let P, P′ be the points, A the angle of reference, PL, P′L′ the perpendiculars to one arm, PM, P′M′ the perpendiculars to the other arm.

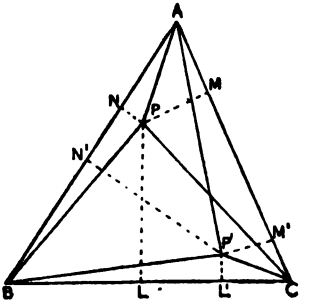
Then the following are evidently pairs of similar triangles, APL, AP′M′, and APM, AP′L′. ∴ PL : P′M′ = PA : P′A, and PM : P′L′ = PA : P′A. ∴ PL : P′M′ = PM : P′L′. ∴ PL.P′L′ = PM.P′M′.

Converse.—If the rectangle contained by the perpendiculars drawn to one of the arms of a given angle from two given points is equal to the rectangle contained by the perpendiculars drawn to the other arm from the same two

points, the straight lines joining the two points to the vertex of the angle are isogonal conjugates with respect to the angle.

This may easily be proved by *reductio ad absurdum.*

(46) THEOREM.—*If three straight lines drawn from the vertices of a triangle are concurrent, their isogonal conjugates with respect to the angles at these vertices are also concurrent.*



Proof. — Let ABC be the triangle and P the point of concurrence of the original three lines. Let the isogonal conjugates of BP and CP meet in P'. Join AP'. From P and P' draw to BC, CA, AB respectively the pairs of perpendiculars PL and P'L', PM and P'M', PN and P'N'.

Then (*Art.* 45) PN.P'N' = PL.P'L' = PM.P'M'. ∴ (*Art.* 45, *conv.*) AP' is the isogonal conjugate of AP with respect to the angle A. ∴ the isogonal conjugates of AP, BP, CP are concurrent in P'.

Def.—Two points related to each other as P and P' are said to be *isogonal conjugates* with respect to the triangle.

(47) THEOREMS.—

(i) *The incentre and excentres of a triangle are their own isogonal conjugates.*

This is obvious.

(ii) *The orthocentre* (H) *and the circumcentre* (O) *of a triangle are isogonal conjugates.*

This follows at once from *addendum C to Euc.* vi.

(iii) *The centroid* (G) *and the symmedian point* (K) *of a triangle are isogonal conjugates.*

Proof.—See figure of *Art.* 44. Let ABC be the triangle and MN an antiparallel to BC meeting CA in M and AB

in N. Let the symmedian AK bisect MN in P, and the median AG bisect BC in A′.

Then since ABC, AMN are similar triangles, AB/BC = AM/MN. But BA′ = ½BC, and MP = ½MN. ∴ AB/BA′ = AM/MP. And angle ABA′ = angle AMP. ∴ ABA′, AMP are similar triangles. ∴ angle BAG = angle CAK. Similarly it may be proved that the connectors of G and K with the other vertices are equally inclined to the arms of the angles at these vertices.

Cor.—The medians and symmedians of a triangle are isogonal conjugates with respect to the angles through whose vertices they pass.

(48) THEOREM.—*The distances of* K *from the sides of the triangle* ABC *are proportional to the sides.*

Proof.—In the figure of *Art.* 44 let x, y, z be the distances of K from the sides a, b, c respectively, and let m, n be the distances of A′ from the sides b, c respectively.

Then $ym = zn$ (*Arts.* 45; 47, iii.); and, since ACA′, ABA′ are equal triangles, $bm = cn$. ∴ $y/b = z/c$. Similarly each $= x/a$.

$$Cor.— \quad \frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{ax+by+cz}{a^2+b^2+c^2} = \frac{2\triangle}{a^2+b^2+c^2}.$$

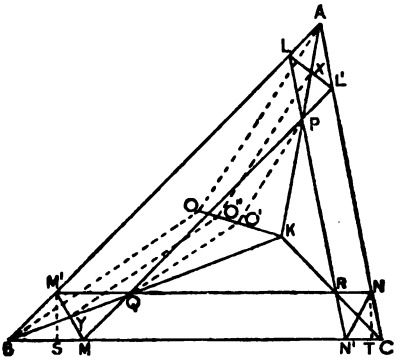
Converse.—If the distances of a point from the sides of a triangle are proportional to the sides, the point is the symmedian point of the triangle.

This may easily be proved by *reductio ad absurdum.*

(49) THEOREM.—*If in a triangle* ABC *whose symmedian point is* K, *points* P, Q, R *be taken on* AK, BK, CK *respectively, so that* PQ, QR, RP *are parallel to* AB, BC, CA *respectively, and if* PQ *meet* CA, BC *in* L′, M *respectively,* QR *meet* AB, CA *in* M′, N *respectively, and* RP *meet* BC, AB *in* N′, L *respectively, then* LL′, MM′, NN′ *are respectively antiparallel to the sides* BC, CA, AB, *and are equal.*

Proof.—ALPL′, BMQM′, CNRN′ are parallelograms. ∴ LL′, MM′, NN′ are bisected by the symmedians, and are therefore antiparallel to BC, CA, AB respectively

(*Art.* 43, v.) Again, from M′, N draw M′S, NT perpendicular to BC. Then angle M′MS = A = angle NN′T; angle M′SM = angle NTN′; and M′S = NT. ∴ triangles



M′SM, NTN′ are congruent. ∴ MM′ = NN′. Similarly each = LL′.

Converse.—If M, N′; N, L′; L, M′ are pairs of points on the sides BC, CA, AB respectively of a triangle ABC, such that LL′, MM′, NN′ are equal and respectively antiparallel to BC, CA, AB, then M′N, N′L, L′M are respectively parallel to BC, CA, AB, and intersect two and two on the symmedians of the triangle.

Proof.—Let M′S, NT be perpendiculars to BC, and let M′N, N′L, L′M intersect in P, Q, R. Then angle M′MS = A = angle NN′T; angle M′SM = angle NTN′; MM′ = NN′. ∴ triangles M′SM, NTN′ are congruent. ∴ M′S = NT. ∴ M′N is parallel to BC. Similarly N′L, L′M are parallel to CA, AB respectively. ∴ ALPL′, BMQM′, CNRN′ are parallelograms. ∴ AP, BQ, CR bisect LL′, MM′, NN′ respectively. ∴ (*Art.* 43, v.) AP, BQ, CR are symmedians.

(50) THEOREM.—*In the figure of the last article* L′, L, M′, M, N′, N *lie on a circle called a* Tucker circle, *whose centre lies on* OK, *and bisects the distance between* O *and the circumcentre of* PQR.

Proof.—Since MM′, NN′, are antiparallel, and N′L, L′M are parallel, to CA, AB respectively, angle M′MN′ = supplement of A = angle NN′M = supplement of N′LM′ = supplement of NL′M. ∴ M, M′, N, N′ are concyclic, the circle which passes through M, M′, N′ also passes through L, and the circle which passes through N, N′, M also passes through L′. ∴ L′, L, M′, M, N′, N are concyclic.

Again, let X, Y be the mid-points of LL′, MM′ respectively. Join OK, OA, OB. Through P draw PO′ perpendicular to LL′, and meeting OK in O′, and at X draw XO″ perpendicular to LL′ and meeting OK in O″. Join O′Q, O″Y. Then OA is perpendicular to LL′ (*Art.* 43, vii. *b*), and is therefore parallel to O′P. ∴ OO′/O′K = AP/PK = BQ/QK (*Euc.* vi. 2). ∴ O′Q is parallel to OB, and is therefore perpendicular to MM′. ∴ since LL′, MM′ are respectively antiparallel to QR, RP with respect to angles RPQ, PQR, the point O′ is the circumcentre of PQR (*Art.* 43, vii. *b*). Again, since OA is parallel to O″X, OO″/O″K = AX/XK = BY/YK. ∴ O″Y is parallel to OB, and is therefore perpendicular to MM′. Thus O″X, O″Y respectively bisect LL′, MM′, chords of the Tucker circle, perpendicularly. ∴ O″ is the centre of the Tucker circle. Finally, since AX = XP, OO″ = O″O′.

(51) *Particular cases of Tucker Circles*—

(i) When the parallels M′N, N′L, L′M coincide with BC, CA, AB respectively, the Tucker circle coincides with the circumcircle.

(ii) When the parallels M′N, N′L, L′M pass through K, the circle is called the *Triplicate Ratio Circle.*

In this case P, Q, R, O′ coincide with K, and the centre O″ is the mid-point of OK.

Triangles KMN′, ABC are similar.

∴ $MN'^2/a^2 = MN'.x/2\triangle$, where x is the distance of K from BC.

∴ $MN' = a^2x/2\triangle$.

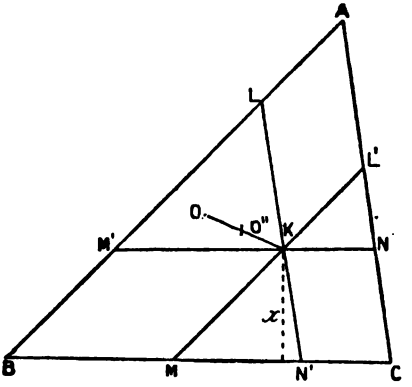
But $x = 2a\triangle/(a^2+b^2+c^2)$. (*Art.* 48.)

∴ $MN' = a^3/(a^2+b^2+c^2)$.

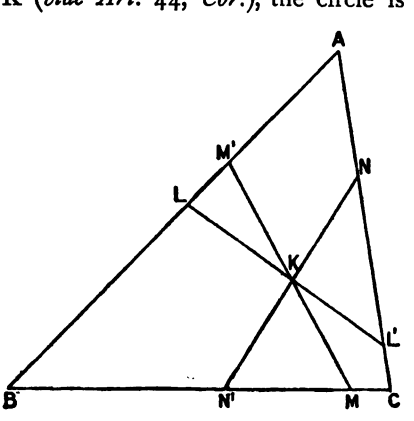
Similarly $NL' = b^3/(a^2+b^2+c^2)$,
and $LM' = c^3/(a^2+b^2+c^2)$.

$\therefore$ $MN' : NL' : LM' = a^3 : b^3 : c^3$.

It is from this property that the circle derives its name.



(iii) When the antiparallels LL', MM', NN' pass through K (*vide Art.* 44, *Cor.*), the circle is called the



Cosine circle. In this case X, Y, and the mid-point of NN' coincide with K, and K is the centre.

The circle derives its name from the following property:—

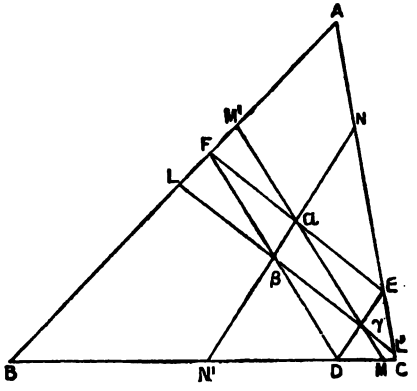
LML′M′, MNM′N′, NLN′L′ are rectangles since their diagonals are equal. Hence if ρ denote the radius of the circle,

$$LM' = LL' \cos M'LL' = 2\rho \cos C.$$

Similarly $MN' = 2\rho \cos A$, and $NL' = 2\rho \cos B$.

$$\therefore MN' : NL' : LM' = \cos A : \cos B : \cos C.$$

(iv) When the antiparallels LL′, MM′, NN′ intersect two and two in the mid-points of the sides of the pedal triangle of ABC, the circle is called the *Taylor circle.*



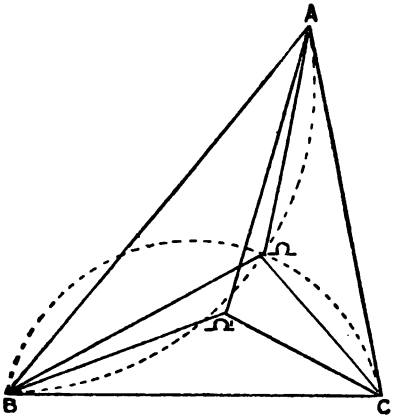
That this is a genuine particular case of a Tucker circle may be shown as follows: Let α, β, γ be the mid-points of the sides EF, FD, DE respectively of the pedal triangle, and let $\beta\gamma$ meet AB, CA in L, L′ respectively, $\gamma\alpha$ meet BC, AB in M, M′ respectively, and $\alpha\beta$ meet CA, BC in N, N′ respectively. Then LL′, MM′, NN′ are respectively parallel to EF, FD, DE. ∴ they are respectively antiparallel to BC, CA, AB (*Art.* 43, vi. *Cor.*)

Again, angle $\alpha MN'$ = angle $\alpha N'M$ (*Art.* 43, viii.) ∴ $\alpha M = \alpha N'$. Similarly $\alpha M' = \alpha F = \alpha E = \alpha N$. ∴ $MM' = NN'$.

Similarly each = LL′. Thus LL′, MM′, NN′ are equal and respectively antiparallel to BC, CA, AB. ∴ L, L′, M, M′, N, N′ lie on a Tucker circle.

Since αMN′ is an isosceles triangle, the perpendicular bisector of the chord MN′ passes through α and bisects angle $\gamma\alpha\beta$. ∴ the centre of the Taylor circle is the in-centre of the triangle $\alpha\beta\gamma$.

(52) (i) PROBLEM.—*To find a point* Ω *connected with a triangle* ABC (*the letters* A, B, C *being arranged counter-clockwise*) *such that angle* ΩBC = *angle* ΩCA = *angle* ΩAB.



Construction.—Describe one circle to pass through A and touch BC at B, and another to pass through B and touch CA at C. Then Ω the second point of intersection of these circles is the point required.

Proof.—Join ΩA, ΩB, ΩC. Then since CA is a tangent to the circle BΩC, and BC is a tangent to the circle AΩB, angle ΩCA = angle ΩBC = angle ΩAB (*Euc.* iii. 32).

Cor.—The circumcircle of triangle CΩA touches AB at A.

Defs.—Ω is called the *positive Brocard point* of triangle

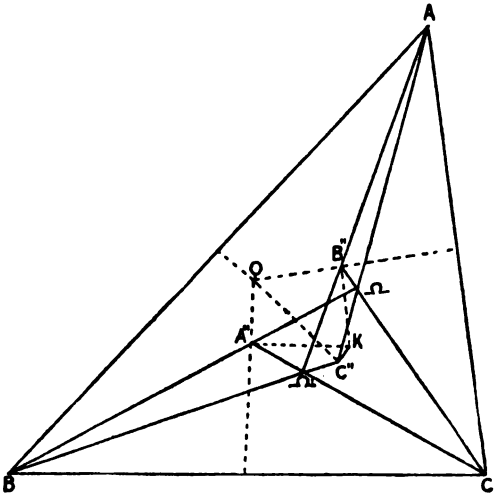
ABC. The isogonal conjugate of Ω, which is denoted by Ω' (and which may be found independently by drawing circles through B to touch CA at A, and through C to touch AB at B), is called the *negative Brocard point* of triangle ABC.

Each of the equal angles ΩBC, $\Omega'BA$, &c., is called the *Brocard angle* of the triangle, and is denoted by ω.

The following theorems are obvious :—

(ii) Angle $B\Omega C$ = supplement of C = angle $C\Omega'A$;
Angle $C\Omega A$ = supplement of A = angle $A\Omega'B$;
Angle $A\Omega B$ = supplement of B = angle $B\Omega'C$.

(53) THEOREM.—*Ω, Ω' lie on the circle having* OK *as diameter.*



Proof.—Let A″, B″, C″ be the respective points of intersection of the pairs of lines $B\Omega$, $C\Omega'$; $C\Omega$, $A\Omega'$; $A\Omega$, $B\Omega'$. Then triangles BA″C, CB″A, AC″B are isosceles and equiangular, each of their base angles being equal to ω. Since the triangles are isosceles, perpendiculars drawn

through A″, B″, C″ to BC, CA, AB respectively will bisect the sides to which they are perpendicular, and will therefore meet in O, the circumcentre of ABC. Since the triangles are equiangular, the angles $\Omega A''\Omega'$, $\Omega B''\Omega'$, $\Omega C''\Omega'$ are each equal or supplementary to 2ω. ∴ A″, B″, C″, Ω, Ω′ lie on a circle. Angle A″OB″ = supplement of C = angle A″ΩB″ (or its supplement). ∴ O lies on the same circle. Through A″, B″ draw parallels to BC, CA respectively, meeting in a point K, and join C″K. Then angle A″KB″ = C (or the supplement of C) = supplement of angle A″ΩB″. ∴ K also lies on the circle. ∴ angle B″KC″ = angle B″Ω′C″ (or its supplement) = supplement of A (or A). ∴ C″K is parallel to AB. Since A″K, B″K, C″K are respectively parallel to BC, CA, AB, the distances of K from BC, CA, AB are equal to the distances of A″, B″, C″ respectively from the same sides. Let these distances be represented by x, y, z. Then, since triangles BA″C, CB″A, AC″B are similar, $x/BC = y/CA = z/AB$. ∴ K is the symmedian point of ABC (*Art.* 48, *Conv.*) Finally, OA″K is a right angle. ∴ OK is the diameter of the circle.

Defs.—The circle having OK as diameter is called the *Brocard circle* of ABC.

A″B″C″ is called the *first Brocard triangle* of ABC.

Cor. i.—The Brocard circle is the circumcircle of the first Brocard triangle.

Cor. ii.—A triangle and its first Brocard triangle are inversely similar.

Cor. iii.—The Brocard circle and the Triplicate Ratio circle of a triangle are concentric.

CHAPTER V.

HARMONIC RANGES AND PENCILS.

(54) *Defs.*—A system of collinear points is called a *range*, and the straight line on which the points lie is called the *axis* of the range.

If A, B, C, D be four collinear points, such that

$$\frac{AC}{CB} = -\frac{AD}{DB},$$

then they are said to form a *harmonic range;* C and D are called *harmonic conjugates* with respect to A and B; and AB is said to be *harmonically divided* in C and D.

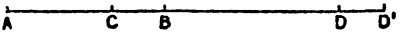
The particular range just defined may be indicated by the expression *harmonic range ABCD* where the order of the letters is significant. In accordance with this notation the expression harmonic range ACBD, for example, implies that $AB/BC = -AD/DC$, and that B, D are harmonic conjugates with respect to A, C.

(55) THEOREM.—*If* ABCD *is a harmonic range, the segment* AB *is divided internally by one of the points* C, D, *and externally by the other.*

Proof.—If AC/CB be positive, C divides AB internally, and AD/DB is negative, so that D divides AB externally (*Art.* 3). Again, if AC/CB be negative, C divides AB externally, and AD/DB is positive, so that D divides AB internally (*Art.* 3).

The theorem may also be stated as follows: If ABCD is a harmonic range, A, B are separated from each other, and C, D are separated from each other (*Art.* 7).

(56) THEOREM.—*A point has only one harmonic conjugate with respect to two given points.*



Proof.—Let D and D′ be harmonic conjugates of C with respect to A and B. Then $AC/CB = -AD/DB$, and $AC/CB = -AD'/D'B$. $\therefore AD/DB = AD'/D'B$. $\therefore$ D and D′ coincide (*Art.* 4).

(57) In the harmonic range ABCD, let A and B be fixed points. Consider, with the aid of *Arts.* 3 and 4, how the positions of C and D are affected by changes in the value of AC/CB.

(i) If $AC/CB = 0$, C coincides with A, and $AD/DB = 0$, so that D also coincides with A.

(ii) If $AC/CB = 1$, C bisects AB internally, and $AD/DB = -1$, so that D is the point at infinity on the line. This is an important theorem, and may be stated thus: A segment of a straight line is divided harmonically by the internal point of bisection and by the point at infinity on the line.

(iii) If $AC/CB = \infty$, C coincides with B, and $AD/DB = \infty$, so that D also coincides with B.

(iv) If $AC/CB > 0$ and $< +1$, $AC < CB$, and $AD/DB < 0$ and > -1, so that D lies on the side of A opposite to B. Thus, in this case, D, A, C lie on the same side of the middle point of AB.

(v) If $AC/CB > +1$ and $< +\infty$, $AC > CB$, and $AD/DB < -1$ and $> -\infty$, so that D lies on the same side of A as B. Thus, in this case, C, B, D lie on the same side of the middle point of AB.

Generally as AC/CB assumes successively all positive values from 0 to ∞, C moves continuously from A to B along the finite segment AB, and D moves continuously from A to B along the *infinite* segment AB; and as AC/CB assumes successively all negative values from ∞ to 0, C moves continuously from B to A along the *infinite* segment

AB, and D moves continuously from B to A along the finite segment AB. It thus appears that C and D make the circuit of the line in *opposite* directions, and that they become coincident in A and B.

(58) THEOREM.—*If* ABCD *is a harmonic range, then* ABDC, BACD, BADC, CDAB, CDBA, DCAB, DCBA *are also harmonic ranges.*

Proof.— $AC/CB = -AD/DB.$

Hence—

(i) $AD/DB = -AC/CB.$ $\therefore$ ABDC is a harmonic range.
(ii) $BC/CA = -BD/DA.$ $\therefore$ BACD " "
(iii) $BD/DA = -BC/CA.$ $\therefore$ BADC " "
(iv) $CA/AD = -CB/BD.$ $\therefore$ CDAB " "
(v) $CB/BD = -CA/AD.$ $\therefore$ CDBA " "
(vi) $DA/AC = -DB/BC.$ $\therefore$ DCAB " "
(vii) $DB/BC = -DA/AC.$ $\therefore$ DCBA " "

Cor.—If C and D are harmonic conjugates with respect to A and B, then A and B are harmonic conjugates with respect to C and D (iv, v, vi, or vii above).

The eight harmonic ranges mentioned in this theorem make together only *one* geometric form, consisting of four collinear points, of which A, B alternate with C, D, and of which A, B and C, D are pairs of conjugates. By an extension of its meaning, the expression *harmonic range ABCD*, previously defined (*Art.* 54) in a more restricted sense, is often used to denote this geometric form.

(59) THEOREM.—*If* ABCD *is a harmonic range, then* AC, AB, AD *are in harmonical progression.**

Proof.— $$\frac{AC}{CB} = -\frac{AD}{DB}.$$

$$\therefore \frac{AC}{AD} = \frac{-CB}{DB} = \frac{AC-AB}{AB-AD}. \quad (\textit{Art.}\ \text{ii, } \textit{Cor.})$$

* In Algebra a, b, c are said to be in harmonical progression, when

$$\frac{a}{c} = \frac{a-b}{b-c}.$$

Cor. i.—If ABCD is a harmonic range, then the following are triads of segments in harmonical progression: BC, BA, BD; CA, CD, CB; DA, DC, DB. This follows from the fact that if ABCD is a harmonic range, BACD, CDAB, DCAB are also harmonic ranges. (*Art.* 58.)

Cor. ii.—If ABCD is a harmonic range, the following groups are in arithmetical progression:—

$$\frac{1}{AC}, \frac{1}{AB}, \frac{1}{AD}; \quad \frac{1}{BC}, \frac{1}{BA}, \frac{1}{BD};$$

$$\frac{1}{CA}, \frac{1}{CD}, \frac{1}{CB}; \quad \frac{1}{DA}, \frac{1}{DC}, \frac{1}{DB}.$$

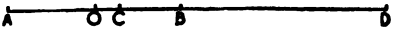
These results may be expressed in the following identities:—

$$\frac{1}{AC}+\frac{1}{AD}=\frac{2}{AB}; \quad \frac{1}{BC}+\frac{1}{BD}=\frac{2}{BA};$$

$$\frac{1}{CA}+\frac{1}{CB}=\frac{2}{CD}; \quad \frac{1}{DA}+\frac{1}{DB}=\frac{2}{DC}.$$

The student should state and prove for himself the converses of the Theorems in this and in the next article.

(60) THEOREM.—*If* O *be the middle point of* AB *in the harmonic range* ABCD, *then* OC, OA (*or* OB), OD *are in geometrical progression.*



Proof.— $$\frac{AC}{CB}=-\frac{AD}{DB}=\frac{AD}{BD};$$

i.e., $$\frac{AO+OC}{OB-OC}=\frac{AO+OD}{OD-OB}.$$

$$\therefore \frac{AO+OC}{AO-OC}=\frac{AO+OD}{OD-AO}.$$

$\therefore$ dividendo and componendo,

$$\frac{2OC}{2AO}=\frac{2AO}{2OD}. \quad \therefore \frac{OC}{OA}=\frac{OA}{OD}.$$

Cor. i.— $OA^2 = OB^2 = OC.OD.$

Cor. ii.— $AD.BD = CD.OD.$

Proof.— $AD.BD = OD^2 - OB^2$ (*Euc.* ii. 6)
$= OD^2 - OC.OD = (OD - OC)OD = CD.OD.$

Cor. iii.— $$\frac{OC}{OD} = \frac{AC^2}{AD^2} = \frac{BC^2}{BD^2}.$$

Proof.— $$\frac{OC}{AO} = \frac{AO}{OD} = \frac{OC + AO}{AO + OD} = \frac{AC}{AD}.$$

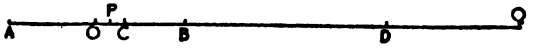
Also $$\frac{OC}{OB} = \frac{OB}{OD} = \frac{OC - OB}{OB - OD} = \frac{BC}{DB}.$$

But, since OC : OD is the duplicate ratio of OC to AO (or OB),

$$\frac{OC}{OD} = \frac{OC^2}{AO^2} = \frac{OC^2}{OB^2} = \frac{AC^2}{AD^2} = \frac{BC^2}{BD^2}.$$

(61) THEOREM.—*If* C, D *and* P, Q *are two pairs of harmonic conjugates with respect to* A *and* B, *and* O *is the middle point of* AB, *then*

$$\frac{PC.PD}{QC.QD} = \frac{PO}{QO} = \frac{AP^2}{AQ^2} = \frac{BP^2}{BQ^2}.$$



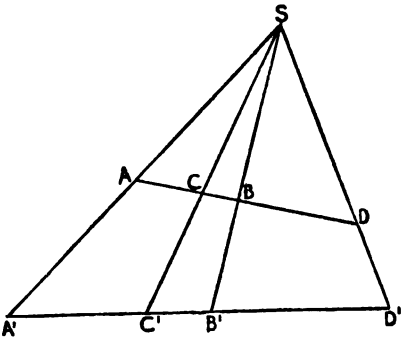
Proof.—$PC.PD = (PO + OC)(PO + OD)$
$= PO^2 + PO.OC + PO.OD + OC.OD.$
But $OC.OD = OB^2 = OP.OQ = PO.QO.$
$\therefore PC.PD = PO^2 + PO.OC + PO.OD + PO.QO$
$= PO(PO + OC + OD + QO).$
Similarly $QC.QD = QO(QO + OC + OD + PO).$
$$\therefore \frac{PC.PD}{QC.QD} = \frac{PO}{QO}.$$

But (*Art.* 60, iii.) $$\frac{PO}{QO} = \frac{AP^2}{AQ^2} = \frac{BP^2}{BQ^2}.$$

(62) *Def.*—A system of concurrent straight lines is

called a *pencil*; the straight lines being called *rays*, and the point of concurrence the *vertex* of the pencil.

THEOREM.—*If a pencil of four rays cuts one transversal in points which form a harmonic range, it cuts all other transversals in points which form harmonic ranges.*



Proof.—Let S be the vertex of the pencil and ABCD the harmonic range in which the pencil is cut by one transversal. Let the rays SA, SC, SB, SD cut any other transversal in the points A′, C′, B′, D′ respectively.

Then, since two triangles, which have an angle of the one equal to an angle of the other, have to each other the ratio compounded of the ratios of their sides (see *Euc.* vi. 23).

$$A'SC' : ASC^{*} = A'S.C'S : AS.CS,$$

and

$$B'SD' : BSD = B'S.D'S : BS.DS.$$

$$\therefore \frac{A'SC'.B'SD'}{ASC.BSD} = \frac{A'S.C'S.B'S.D'S}{AS.CS.BS.DS}.$$

Similarly

$$\frac{A'SD'.C'SB'}{ASD.CSB} = \frac{A'S.D'S.C'S.B'S}{AS.DS.CS.BS}.$$

* ASC denotes *triangle* ASC.

$$\therefore \frac{A'SC'.B'SD'}{ASC.BSD} = \frac{A'SD'.C'SB'}{ASD.CSB}.$$

$$\therefore \frac{A'SC'}{C'SB'} \cdot \frac{ASD}{BSD} = \frac{A'SD'}{B'SD'} \cdot \frac{ASC}{CSB}.$$

$$\therefore \frac{A'C'}{C'B'} \cdot \frac{AD}{BD} = \frac{A'D'}{B'D'} \cdot \frac{AC}{CB}.$$

But
$$\frac{AD}{BD} = \frac{AC}{CB}.$$

$$\therefore \frac{A'C'}{C'B'} = \frac{A'D'}{B'D'} = -\frac{A'D'}{D'B'}.$$

(63) *Def.* — If SA, SB, SC, SD be four concurrent straight lines such that any transversal cuts them in the harmonic range ABCD, then they are said to form a *harmonic pencil;* SC and SD are called *harmonic conjugates* with respect to SA and SB; and the angle ASB is said to be *harmonically divided* by SC and SD. The particular pencil just defined may be indicated by the expression *harmonic pencil S.ABCD*, where the order of the letters is significant.

The following theorems regarding harmonic pencils are directly deducible from the corresponding theorems regarding harmonic ranges :—

(i) If S.ABCD is a harmonic pencil, then the angle ASB is divided internally by one of the rays SC, SD, and externally by the other. Otherwise: SA, SB are separated from each other, and SC, SD are separated from each other (*Art.* 55).

(ii) A ray has only *one* harmonic conjugate with respect to two given rays (*Art.* 56).

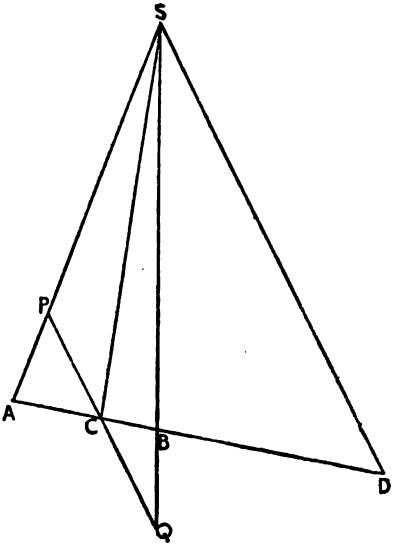
(iii) If SC, SD be two straight lines revolving round S in such a manner that they are always harmonic conjugates to SA, SB, they revolve in *opposite directions*, both at the same time approaching or receding from SA (or SB), and they become coincident in SA and SB (*Art.* 57).

(iv) If S.ABCD is a harmonic pencil, then S.ABDC, S.BACD, S.BADC, S.CDAB, S.CDBA, S.DCAB, S.DCBA are also harmonic pencils.

If SC, SD are harmonic conjugates with respect to SA, SB, then SA, SB are harmonic conjugates with respect to SC, SD (*Art.* 58).

The eight harmonic pencils above mentioned make together only *one* geometric form, consisting of four concurrent straight lines, of which SA, SB alternate with SC, SD, and of which SA, SB and SC, SD are pairs of conjugates. By an extension of its meaning, the expression, *harmonic pencil S.ABCD*, previously defined in a more restricted sense, is often used to denote this geometric form.

(64) THEOREM.—*If a transversal be drawn parallel to any ray of a harmonic pencil, the conjugate ray bisects the*



segment of the transversal intercepted between the remaining two rays.

Proof.—The pencil cuts every transversal, and therefore the transversal in question, in points which form a harmonic range (*Art.* 62). But since parallel straight lines meet at infinity, one of these points is at infinity. Therefore its conjugate bisects the segment connecting the remaining two points (*Art.* 57, ii.)

The theorem may be proved independently of *Art.* 62, as follows:—

Let S.ABCD be the harmonic pencil, and PCQ a transversal parallel to SD, and cutting SA, SB, SC in P, Q, C respectively. Through C draw a transversal, cutting the pencil in the harmonic range ABCD. Then AC : CB = AD : BD. ∴ AC : AD = CB : BD. But, by similar triangles, AC : AD = PC : SD, and CB : BD = CQ : SD. ∴ PC : SD = CQ : SD. ∴ PC = CQ.

The student should vary the figure and the proof by taking PQ parallel to other rays.

Converse.—If a transversal parallel to one ray of a pencil of four rays is bisected by the alternate ray, the pencil is harmonic.

Proof.—The transversal is cut in four points (one of them at infinity) which form a harmonic range (*Art.* 57, ii.) Therefore (*Art.* 63, *Def.*) the pencil is harmonic.

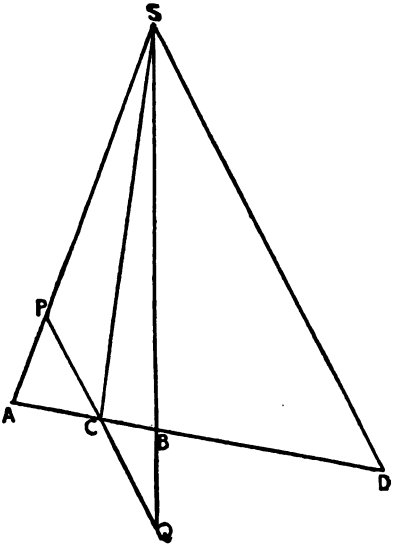
A proof similar to the alternative proof of the theorem might be given.

(65) PROBLEM.—*Given three points of a harmonic range or three rays of a harmonic pencil, to find the fourth point or ray.*

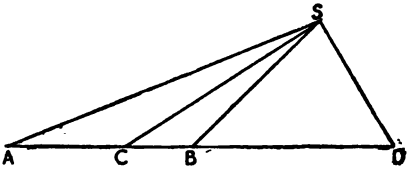
(i) Let A, B, C be the three given points, A and B being regarded as conjugates. Through C draw a straight line, and cut off on each side of C equal lengths PC, CQ. Join PA and QB, and let PA, QB meet in S. Through S draw a parallel to PQ meeting the axis of the range in D. Then (*Art.* 64, *Converse*) S.ABCD is a harmonic pencil. Therefore ABCD is a harmonic range.

(ii) Let SC, SD, SA be the three given rays, SC and SD being taken as conjugates. Through C any point on SC draw a parallel to SD cutting SA in P. Produce PC

to Q, making CQ equal to PC. Join SQ. Then (*Art.* 64, *Converse*) S.CDAB is a harmonic pencil.



(66) THEOREM.—*The arms of any angle* ASB *form with its internal and external bisectors*, SC, SD, *a harmonic pencil.*



Proof.—Let ACBD be a transversal cutting SA, SC, SB, SD.

Then (*Euc.* vi. 3 and *Cor.*)

$$AC : CB = AS : SB = AD : BD.$$

$$\therefore AC/CB = -AD/DB.$$

$\therefore$ ABCD is a harmonic range.

$\therefore$ S.ABCD is a harmonic pencil.

Cor.—The arms of a right angle and two straight lines through the vertex equally inclined to one of the arms of the right angle, form a harmonic pencil.

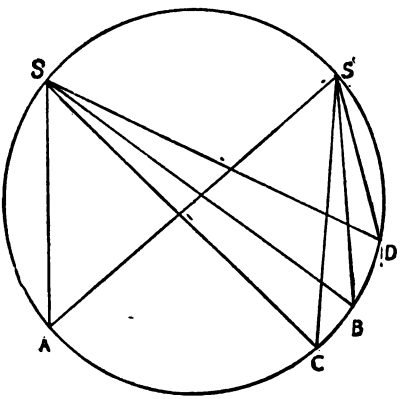
Converse i.—If the angle between two conjugate rays of a harmonic pencil is bisected internally or externally by one of the other rays, it is also bisected externally or internally by the remaining ray.

Proof.—SC (or SD) is a harmonic conjugate, and therefore (*Art.* 63, ii.) the only harmonic conjugate of SD (or SC) with respect to SA, AB.

Converse ii.—If in a harmonic pencil two conjugate rays form a right angle, the remaining two rays are equally inclined to either of the arms of the angle.

Proof.—SA (or SB) is a harmonic conjugate, and therefore (*Art.* 63, ii.) the only harmonic conjugate of SB (or SA) with respect to SC, SD.

(67) THEOREM.—*If* A, B, C, D *are four points on the*



circumference of a circle such that the straight lines joining them to a certain point, S, *on the circumference form a*

harmonic pencil, then the straight lines joining them to any *point*, S′, *on the circumference, form a harmonic pencil.*

Proof.—The angles subtended by arcs AC, CB, BD at S and S′ are equal. Therefore the two pencils are superposable. Therefore they possess the same properties.

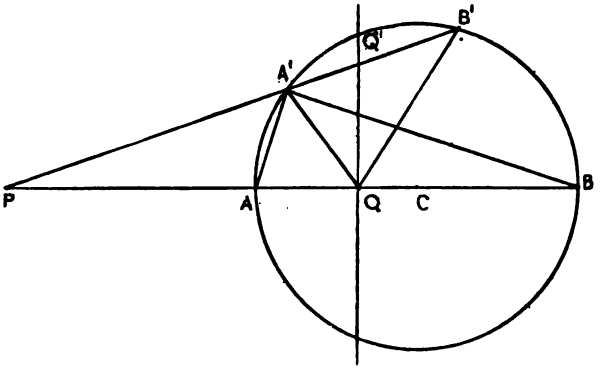
Cor.—The straight lines joining any one of the points A, B, C, D to the remaining three, together with the tangent at the point, form a harmonic pencil.

CHAPTER VI.

POLES AND POLARS. THE PRINCIPLE OF DUALITY.

(68) *Defs.*—If P be a fixed point inside or outside a fixed circle, and if through P a variable chord be drawn cutting the circle in A and B, and if Q be the harmonic conjugate of P with respect to A and B, then the locus of Q is called the *polar* of P, and P is called the *pole* of the locus of Q.

THEOREM.—*The polar of a point with reference to a circle is a straight line perpendicular to the diameter which passes through the point.*



Proof.—Let P be any point inside or outside a circle, PAB the diameter through P, Q the harmonic conjugate of

P with respect to A and B, PA'B' any chord through P, and Q' the harmonic conjugate of P with respect to A' and B'. Join QQ', QA', QB', A'A, and A'B.

Then, since PQAB is a harmonic range, A'.PQAB is a harmonic pencil. But AA'B is a right angle. ∴ (*Art.* 66, ii.) A'A bisects the angle PA'Q. ∴ PA'/A'Q = PA/AQ. The same result is true if for A' we put any other point on the circumference of the circle. Thus PB'/B'Q = PA/AQ. ∴ PA'/A'Q = PB'/B'Q. ∴ PA'/PB' = A'Q/B'Q. ∴ QP bisects the angle A'QB' internally or externally. Therefore, since PQ'A'B' is a harmonic range, and consequently Q.PQ'A'B' a harmonic pencil, PQQ' is a right angle (*Art.* 66, i.) Therefore the locus of Q' is a straight line perpendicular to AB.

Cor. i.—If C is the centre of the circle, the polar of P is the straight line perpendicular to PC through a point Q which is determined by the condition $CP.CQ = CA^2$.

Proof.—ABPQ is a harmonic range, and C is the middle point of AB. ∴ $CP.CQ = CA^2$.

Cor. ii.—The polar of a point inside the circle lies wholly outside the circle; the polar of a point outside the circle cuts the circle.

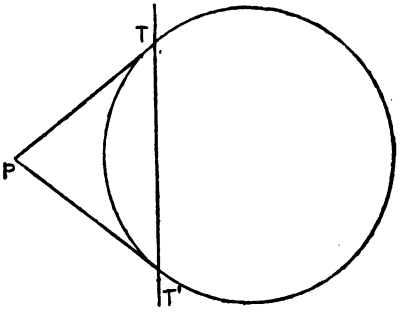
The polar of a point outside the circle need not be considered as lying wholly within the circle; it may be indefinitely prolonged outside the circle. In this case straight lines joining the pole to points on the polar which lie outside the circle may be regarded as cutting the circle in *imaginary* points (*vide Art.* 13).

Cor. iii.—The polar of a point on the circumference of the circle, with respect to the circle, is the tangent at the point; and the pole of a tangent, with respect to the circle, is the point of contact.

(69) THEOREM.—*If from a point* P *outside a circle two tangents* PT, PT' *be drawn to the circle, the chord of contact* TT' *is the polar of* P *with respect to the circle.*

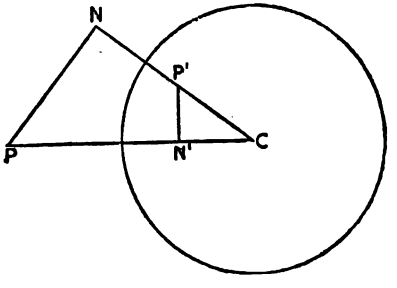
Proof.—PT meets the circle in two points which coincide at T (*Art.* 13). T is the harmonic conjugate of P with

respect to these points (*Art.* 57, iii.) Therefore T lies on the polar of P (*Art.* 68, *def.*) Similarly T′ lies on the polar of P. Therefore, since the polar is a straight line, TT′ is the polar of P.



Cor.—A chord drawn through P a point outside a circle is divided harmonically by P and the chord of contact of the tangents drawn from P.

(70) THEOREM.—*If the polar of a point passes through a second point, the polar of the second point passes through the first.*



Proof.—Let the polar of P with respect to a circle of which C is the centre, pass through P′. Join PC, P′C.

Draw PN perpendicular to CP′, and P′N′ perpendicular to CP. Then P, P′, N, N′ are concyclic. ∴ CP.CN′ = CP′.CN. But, since P′N′ passes through P′ and is perpendicular to CP, it is the polar of P (*Art.* 68). ∴ CP.CN′ = (radius)² (*Art.* 68, i.) ∴ CP′.CN = (radius)². ∴ PN, since it passes through N and is perpendicular to CP′, is the polar of P′ (*Art.* 68, i.); and it passes through P.

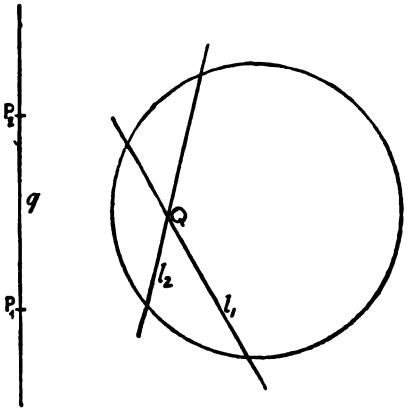
(71) THEOREM.—*If the pole of one straight line lies on a second straight line, the pole of the second straight line lies on the first.*

Proof.—From C, the centre of the circle of reference (*vide* figure of last article), drop a perpendicular CN on the first straight line, and let it meet the second straight line in P′. From C drop a perpendicular CN′ on the second straight line, and let it meet the first straight line in P. Then P, P′, N, N′ are concyclic. ∴ CP.CN′ = CP′.CN. But P′, since it lies on the second straight line and also on the diameter perpendicular to the first straight line, is the pole of the first straight line. ∴ CP′.CN = (radius)². ∴ CP.CN′ = (radius)². ∴ P, since it lies on the diameter perpendicular to the second straight line, is the pole of the second straight line (*Art.* 68, i.); and it lies on the first straight line.

(72) THEOREM.—*The poles of the rays of a pencil form a range the axis of which is the polar of the vertex of the pencil.*

Proof.—Let l_1, l_2, &c., be straight lines intersecting at Q, and let P_1, P_2, &c., be their respective poles with respect to a circle.

Then, since the polar of P_1 passes through Q, the polar of Q passes through P_1. Similarly the polar of Q passes through P_2. Similarly the polar of Q passes through the poles of all the other straight lines which pass through Q. ∴ P_1, P_2, &c., lie on a straight line which is the polar of Q.



Cor.—If a variable chord be drawn to a fixed circle through a fixed point, the locus of the point of intersection of the tangents drawn from the ends of the chord is a straight line—viz., the polar of the fixed point (*Art.* 69).

(73) THEOREM.—*The polars of the points of a range form a pencil the vertex of which is the pole of the axis of the range.*

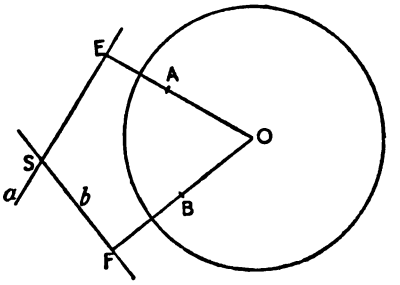
Proof.—(*Vide* figure of last article.) Let P_1, P_2, &c., be points lying on the straight line q, and let l_1, l_2, &c., be their respective polars.

Then, since the pole of l_1 lies on q, the pole of q lies on l_1. Similarly the pole of q lies on l_2. Similarly the pole of q lies on the polars of all the other points which lie on q. $\therefore$ l_1, l_2, &c., intersect at a point which is the pole of q.

Cor.—If from a variable point on a fixed straight line tangents be drawn to a fixed circle, the chord of contact passes through a fixed point—viz., the pole of the fixed straight line (*Art.* 69).

(74) THEOREM.—*Two intersecting straight lines are in-*

clined to each other at the same angle as the straight lines joining their poles to the centre of the circle of reference.



Proof.—Let *a*, *b* be two straight lines intersecting at S, and A, B their respective poles with reference to the circle whose centre is O. Let the connectors OA, OB meet *a*, *b* respectively in E, F. Then (*Art.* 68) the angles at E, F are right. Therefore S, O, E, F are concyclic. Therefore the angles AOB, ESF are either equal or supplementary.

Cor. i.—A pencil is superposable with the pencil formed by joining the poles of its rays to the centre of the circle of reference.

Cor. ii.—The poles of the rays of a harmonic pencil form a harmonic range; and the polars of the points of a harmonic range form a harmonic pencil.

Proof.—By *Cor.* ii. the pencil is superposable with another pencil of which the range is a section. Therefore if the pencil is harmonic the range is harmonic, and conversely.

(75) *Principle of Duality.*—Let F be a figure consisting of points and straight lines. By taking the polar of every point, and the pole of every straight line in F, with respect to a circle, we obtain a second figure, F′, consisting of straight lines and points. F and F′ are called *polar reciprocals.* The expression *reciprocal* is used because we can obtain F from F′ by reversing the process, taking the

pole of every straight line, and the polar of every point in F′ with respect to the circle.

From the theorems proved regarding poles and polars, it is evident that any element of F mentioned in the left-hand column below corresponds to the element of F′ set down directly opposite in the right-hand column.

Elements which may belong to F.	*Correlative elements of F′.*
Point.	Straight line.
Straight line.	Point.
Point of intersection of two straight lines.	Straight line joining two points (*Art.* 72).
Straight line joining two points.	Point of intersection of two straight lines (*Art.* 73).
Range and its axis.	Pencil and its vertex (*Art.* 73).
Pencil and its vertex.	Range and its axis (*Art.* 72).
Harmonic range.	Harmonic pencil (*Art.* 74, ii.)
Harmonic pencil.	Harmonic range (*Art.* 74, ii.)
Point on the circle of reference.	Tangent to the circle of reference (the tangent at the point) (*Art.* 68, iii.)
Tangent to the circle of reference.	Point on the circle of reference (the point of contact) (*Art.* 68, iii.)
Point of intersection of two tangents to the circle of reference.	Chord (the chord of contact) joining two points on the circle of reference (the points of contact) (*Art.* 69).
Chord joining two points on the circle of reference.	Point of intersection of two tangents to the circle of reference (the tangents at the points) (*Art.* 69).
&c.	&c.

We are thus led to the following two important conclusions:—

(i) If a statement affirming that points lie on straight lines, that straight lines pass through points, or

that particular ranges or pencils are harmonic, is true for a figure F, it is also true, the elements having been interchanged as above, for the polar reciprocal of F.

(ii) If a statement affirming that points lie on straight lines, that straight lines pass through points, or that particular ranges or pencils are harmonic, is generally true for *all figures of the same species as the figure F*, it is also true, the elements having been interchanged as above, for *all figures of the same species as the polar reciprocal of F*.

When F or any part of it is inscribed in or circumscribed to a circle, this circle should be taken as the circle of reference; otherwise the circle of reference need not be taken into account. The following is an illustration of conclusion i. :—

Suppose that for a particular octagon ABCDEFGH inscribed in a circle, it is true that the diagonals AE, BF, CG, DH are concurrent. The polar reciprocal of the octagon is another octagon formed by drawing tangents a, b, c, d, e, f, g, h, at the points A, B, C, D, E, F, G, H respectively, and the correlative statement which is true for this second octagon is that the points of intersection of the pairs of sides, a, e; b, f; c, g; d, h, are collinear. It will be observed that, as the statement regarding the concurrency of the diagonals applies only to a particular octagon, the correlative statement regarding the collinearity of the intersections of sides applies only to the actual polar reciprocal of that octagon.

Conclusion ii. is a partial statement of a highly important and most interesting geometrical law, *the principle of reciprocity or duality*. In accordance with it, propositions respecting the *position* of points and lines fall into two groups, each group consisting of statements which are correlative or reciprocal to the statements in the other. Such correlative propositions are usually printed opposite to each other in parallel columns. Sometimes they are combined in a single statement, the words which require to be sub-

stituted in the reciprocal proposition being printed below the corresponding words of the original proposition. The following are a few simple illustrations :—

Two points in a plane determine a straight line, the connector.	Two straight lines in a plane determine a point, the point of intersection.
Three non-collinear points in a plane connect two and two by three straight lines, forming a triangle.	Three non-concurrent straight lines in a plane intersect two and two in three points, forming a trilateral (three-side, triangle).
Theorem of *Art.* 70.	Theorem of *Art.* 71.
Theorem of *Art.* 72.	Theorem of *Art.* 73.
Cor. of *Art.* 72.	*Cor.* of *Art.* 73.
The points of intersection of a straight line with the rays of a harmonic pencil form a harmonic range.	The connectors of a point with the points of a harmonic range form a harmonic pencil.

If four $\left\{\begin{matrix}\text{points}\\\text{tangents}\end{matrix}\right\}$ be taken $\left\{\begin{matrix}\text{on}\\\text{to}\end{matrix}\right\}$ a circle, such that their $\left\{\begin{matrix}\text{connectors}\\\text{intersections}\end{matrix}\right\}$ with a certain fifth $\left\{\begin{matrix}\text{point on}\\\text{tangent to}\end{matrix}\right\}$ the circle form a harmonic $\left\{\begin{matrix}\text{pencil}\\\text{range}\end{matrix}\right\}$, their $\left\{\begin{matrix}\text{connectors}\\\text{intersections}\end{matrix}\right\}$ with every $\left\{\begin{matrix}\text{point on}\\\text{tangent to}\end{matrix}\right\}$ the circle form a harmonic $\left\{\begin{matrix}\text{pencil}\\\text{range}\end{matrix}\right\}$.

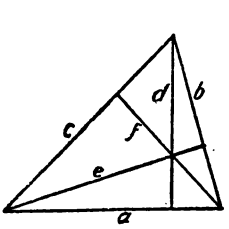
When we have established a theorem of position, we are warranted in assuming the truth of its reciprocal without further proof. As a rule, however, in this book, each of a pair of reciprocal theorems will be separately proved. The process by which we pass from a theorem to its reciprocal,

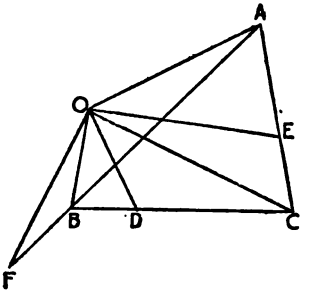
merely by interchanging the expressions, *point* and *straight line*, *concurrent* and *collinear*, *inscribed* and *circumscribed*, &c., is called *reciprocation*. The student is strongly recommended to effect the operation for himself before looking at the reciprocals given in the text.

It must be emphasised that the principle of duality, as above stated, applies only to propositions regarding the *position* of points and straight lines. Considerations regarding the *magnitude* of lines and angles are excluded. The reason for this is that although two points reciprocate into two straight lines, and it may be said that the linear segment connecting the two points corresponds to the angle between the two lines, the *length* of the segment bears no fixed relation to the *size* of the angle. For example, three collinear points reciprocate into three concurrent straight lines, but if one of the points bisects the segment lying between the other two, it does not follow that one of the straight lines bisects the angle between the other two. It is true that two straight lines intersecting at a certain angle have as their reciprocal two points, the connector of which subtends an equal angle at the centre of the circle of reference. But taking the centre of the circle of reference into account complicates the statement of the reciprocal proposition. Thus the law of reciprocity is of partial and difficult application in the geometry of *magnitude*, while it is of universal and easy application in the geometry of pure *position*. Propositions which involve constructions for which compasses or a graduated ruler are necessary, involve magnitude, and are therefore not well adapted, if at all, to reciprocation. Consequently, for the sake of simplicity and to secure generality, the law of reciprocity has been so stated that it applies exclusively to propositions which involve merely such constructions (apart from the circle of reference) as can be made by means of the ungraduated ruler only. The fact that harmonic ranges and pencils, also tangents and their points of contact, admit of reciprocation in the sense of the present article, forms no exception to this statement, for, as will be shown in the next chapter,

these figures can be constructed by means of the ungraduated ruler only.

The following is given as an example of the reciprocation of a theorem in which the magnitude of certain angles is taken into account.





a, *b*, *c* are the sides of a triangle; *d*, *e*, *f* are straight lines through the vertices perpendicular to the opposite sides *a*, *b*, *c* respectively. *d*, *e*, *f* shall be concurrent.

A, B, C are the vertices of a triangle; D, E, F are points on the sides such that their distances from the opposite vertices A, B, C respectively subtend each a right angle at a point O (the centre of the circle of reference). D, E, F shall be collinear.

CHAPTER VII.

HARMONIC RANGES AND PENCILS, AND POLES AND POLARS—*continued.*

(76) *Def.*—Any $\left\{\begin{matrix}\text{point}\\\text{ray}\end{matrix}\right\}$ of a harmonic $\left\{\begin{matrix}\text{range}\\\text{pencil}\end{matrix}\right\}$, its harmonic conjugate, any one of the two remaining $\left\{\begin{matrix}\text{points}\\\text{rays}\end{matrix}\right\}$, and its harmonic conjugate, may be said to *correspond*, respectively, to any $\left\{\begin{matrix}\text{point}\\\text{ray}\end{matrix}\right\}$ of another harmonic $\left\{\begin{matrix}\text{range}\\\text{pencil}\end{matrix}\right\}$, its harmonic conjugate, any one of the two remaining $\left\{\begin{matrix}\text{points}\\\text{rays}\end{matrix}\right\}$, and its harmonic conjugate.

Thus—

If ABCD, A′B′C′D′ be two harmonic ranges, the points A, B, C, D, may be said to correspond, respectively, to the points A′, B′, C′, D′, or A′, B′, D′, C′, or B′, A′, C′, D′, or B′, A′, D′, C′, or C′, D′, A′, B′, or C′, D′, B′, A′, or D′, C′, A′, B′, or D′, C′, B′, A′.

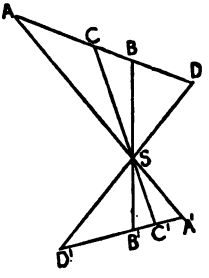
If *abcd*, *a′b′c′d′* be two harmonic pencils, the rays *a*, *b*, *c*, *d*, may be said to correspond, respectively, to the rays *a′*, *b′*, *c′*, *d′*, or *a′*, *b′*, *d′*, *c′*, or *b′*, *a′*, *c′*, *d′*, or *b′*, *a′*, *d′*, *c′*, or *c′*, *d′*, *a′*, *b′*, or *c′*, *d′*, *b′*, *a′*, or *d′*, *c′*, *a′*, *b′*, or *d′*, *c′*, *b′*, *a′*.

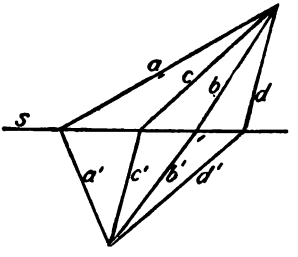
There are thus *eight* different ways in which the $\left\{\begin{matrix}\text{points}\\\text{rays}\end{matrix}\right\}$ of one harmonic $\left\{\begin{matrix}\text{range}\\\text{pencil}\end{matrix}\right\}$ correspond to the $\left\{\begin{matrix}\text{points}\\\text{rays}\end{matrix}\right\}$ of another harmonic $\left\{\begin{matrix}\text{range}\\\text{pencil}\end{matrix}\right\}$. (*Arts.* 58 and 63, iv.)

(77) THEOREMS.—

(*a*) *If the straight lines connecting three pairs of corresponding points of two harmonic ranges* (*not having the same axis*) *meet in a point, the connector of the remaining pair passes through the same point.*

(*a'*) *If the points of intersection of three pairs of corresponding rays of two harmonic pencils* (*not having the same vertex*) *lie on a straight line, the point of intersection of the remaining pair lies on the same straight line.*





Proof.—

Let ABCD, A'B'C'D' be the two harmonic ranges, and let the connectors AA', BB', CC' meet in the point S. Then, since CC' has only one harmonic conjugate with respect to AA', BB' (*Art.* 63, ii.), SD and SD' coincide.

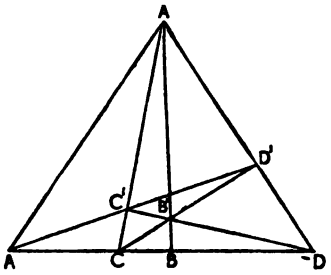
Let *abcd*, *a'b'c'd'* be the two harmonic pencils, and let the points of intersection *aa'*, *bb'*, *cc'* * lie on the straight line *s*. Then, since *cc'* has only one harmonic conjugate with respect to *aa'*, *bb'* (*Art.* 56), *sd* and *sd'* coincide.

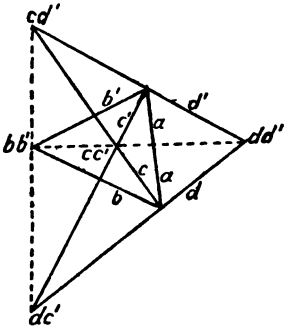
* Just as the connector of two points A, A' is represented by AA', so the point of intersection of two straight lines *a*, *a'* may be represented by *aa'*.

(78) THEOREMS.—

(*a*) *If* ABCD, AB′C′D′ *are two harmonic ranges having a common point* A, *the connector* BB′ *is concurrent with the connectors* CC′, DD′, *and also with the connectors* CD′, DC′.

(*a*′) *If* abcd, ab′c′d′ *are two harmonic pencils having a common ray* a, *the point of intersection* bb′ *is collinear with the points of intersection* cc′, dd′, *and also with the points of intersection* cd′, dc′.





Proof.—

Any straight line through A can be regarded as a connector of the point A of the range ABCD with the corresponding point A of the range AB′C′D′. Therefore the connectors of A and A, C and C′, D and D′, meet in a point. Therefore the connector BB′ passes through the same point (*Art.* 77 *a*). Similarly the connectors of A and A, C and D′, D and C′, meet in a point. Therefore the connector BB′ passes through the same point (*Art.* 77 *a*).

Any point on *a* can be regarded as the point of intersection of the ray *a* of the pencil *abcd* with the corresponding ray *a* of the pencil *ab′c′d′*. Therefore the points of intersection of *a* and *a*, *c* and *c′*, *d* and *d′*, lie on a straight line. Therefore the point of intersection *bb′* lies on the same straight line (*Art.* 77 *a*′). Similarly the points of intersection of *a* and *a*, *c* and *d′*, *d* and *c′*, lie on a straight line. Therefore the point of intersection *bb′* lies on the same straight line (*Art.* 77 *a*′).

(79) *Definitions.*—

(*a*) A system of four straight lines (no three of which are concurrent), and their six points of intersection, is called a *complete quadrilateral.*

The four lines are called the *sides* of the quadrilateral; and the six points the *vertices.*

Two vertices which do not lie on the same side are called *opposite vertices.*

The three connectors of opposite vertices are called *diagonal lines;* and the trilateral (three-side, triangle) determined by the diagonal lines is called the *diagonal triangle* of the quadrilateral.

(*a'*) A system of four points (no three of which are collinear), and their six connectors, is called a *complete quadrangle.*

The four points are called the *vertices* of the quadrangle; and the six connectors the *sides.*

Two sides which do not pass through the same vertex are called *opposite sides.*

The three points of intersection of opposite sides are called *diagonal points;* and the triangle determined by the diagonal points is called the *diagonal triangle* of the quadrangle.

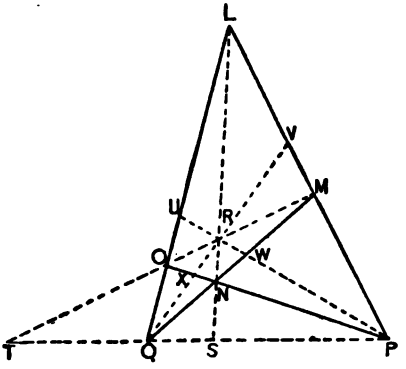
(80) THEOREMS.—

(*a*) *In a complete quadrilateral each pair of angular points of the diagonal triangle forms, with the pair of opposite vertices which lie in the same diagonal line, a harmonic range.*

(*a'*) *In a complete quadrangle each pair of sides of the diagonal triangle forms with the pair of opposite sides which pass through the same diagonal point a harmonic pencil.*

Proof of (*a*).—Let L, M, N, O, P, Q be the six vertices of a complete quadrilateral, and RST its diagonal triangle, the letters being so arranged in the figure that L, N, R, S; O, M, T, R; P, Q, S, T are quartets of collinear points. Let S′ be the harmonic conjugate of R with respect to L and N; and let T′ be the harmonic conjugate of R with respect to O and M. Then since the harmonic ranges

RS'LN, RT'OM have the point R common, S'T' is concurrent with LO, NM, and with LM, NO (*Art.* 78 *a*). Therefore S', T', Q, P are collinear. Therefore S' coincides with S, and T' coincides with T. Therefore RSLN and RTOM are harmonic ranges. Again, since RTOM is a harmonic range, L.RTOM is a harmonic pencil. Therefore STQP, which crosses the pencil, is also a harmonic range.



Proof of a' (reciprocal).—Let L, M, N, O in the above figure be the four vertices of a complete quadrangle. Then PQR is its diagonal triangle. Let PR' be the harmonic conjugate of PQ with respect to PN and PL; and let QR', meeting PR' in R', be the harmonic conjugate of QP with respect to QN and QL. Then since the harmonic pencils P.SR'NL, Q.SR'NL have the ray PQ common, the pairs of rays PN, QN; PL, QL; PR', QR' intersect collinearly (*Art.* 78 *a'*). Therefore R' lies on LN. Also the pairs of rays PN, QL; PL, QN; PR', QR' intersect collinearly (*Art.* 78 *a'*). Therefore R' lies on OM. Therefore R' coincides with R. Therefore P.SRNL and Q.SRNL are harmonic pencils. Again, since P.SRNL is a harmonic pencil, QUOL, where U is the point where PR meets OL, is a harmonic range. Therefore R.QUOL—*i.e.*, R.QPTS—is also a harmonic pencil.

The above two proofs are readily seen to be reciprocal in character; their formal presentation, however, as reciprocals, would have required in a' the use of single letters for lines, and double letters for points,—a somewhat confusing notation where there are many lines in a figure.

Second proof of a'.—Since RSLN, RTOM, STQP are harmonic ranges (a), P.RSLN, Q.RTOM, R.STQP are harmonic pencils.

Cors. i.—

(a) In a complete quadrilateral the pencil formed at any vertex by two sides, a diagonal line, and the connector of the vertex with the point of intersection of the two other diagonal lines, is harmonic.

(a') In a complete quadrangle the range formed on any side by two vertices, a diagonal point, and the point of intersection of the side with the connector of the two other diagonal points, is harmonic.

Thus, if U, W be the points of intersection of PR with LO, MN respectively, and X, V the points of intersection of QR with ON, LM respectively:—

L.PQST, M.LNRS, N.MORT, O.LNRS, P.OMTR, Q.NLSR, are harmonic pencils.

LNRS, MLPV, MORT, NMQW, NOPX, OLQU, are harmonic ranges.

Cors. ii.—

(a) In a complete quadrilateral any two diagonal lines, and the straight lines joining their point of intersection to the two vertices which do not lie on them, form a harmonic pencil.

Thus, R.STPQ, S.RTMO, T.RSLN are harmonic pencils.

(a') In a complete quadrangle any two diagonal points, and the points where the straight line joining them cuts the two sides which do not pass through them, form a harmonic range.

Thus, PQST, QRXV, RPUW are harmonic ranges.

Cor. iii.—The following are triads of concurrent straight lines: VS, PR, MN; VU, MO, PQ; US, QR, ON; RP, TX, MN; UT, QR, LM; UX, LN, QP; OV, LX, UR; NV, MX, WR; NT, MS, WP; OS, LT, UP; SX, TV, PR; NT, OS, XQ; NU, OW, XR; MU, LW, VR; SW, QR, TU; MS, LT, VQ; VW, LN, PQ; VT, PR, LO; RQ, TW, ON; XW, OM, PQ; SX, RP, LO; QR, ML, WS; XW, VU, QP; XU, VW, QP.

Proof.—LMVP, LNRS are harmonic ranges having the point L common. Therefore (*Art.* 78 *a*) VS, PR, MN are concurrent. A similar proof holds for the other triads.

Cor. iv.—The following are triads of collinear points: V, W, S; V, U, T; U, X, S; W, X, T.

The collinearity is involved in the concurrency of the first four triads in *Cor.* iii.

Cor. v.—PQR is the diagonal triangle of the complete quadrilateral whose sides are UV, VW, WX, XU; and RST is the diagonal triangle of the complete quadrangle whose vertices are U, V, W, and X.

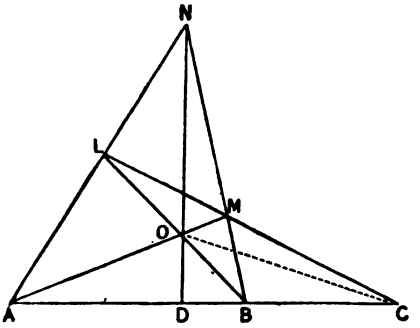
(81) PROBLEMS.—

(*a*) *Given two harmonic conjugates, and another point of a harmonic range, to construct the fourth point, by means of the ungraduated ruler only.*

(*a'*) *Given two harmonic conjugates, and another ray of a harmonic pencil, to construct the fourth ray, by means of the ungraduated ruler only.*

Construction for a.—Let A and B be the harmonic conjugates, and C the other point. Through C draw any straight line. Take L and M any two points on this line. Join AL, BM, and let the joins meet in N. Join AM, BL, and let the joins meet in O. Join NO, and let the join meet the axis of the range in D. D is the point required.

Proof.—The figure is that of a complete quadrilateral, of which A and B are opposite vertices, and C and D vertices of the diagonal triangle. Therefore ABCD is a harmonic range (*Art.* 80 *a*).



Construction for a′.—Let OA and OB, in the above figure, be the harmonic conjugates, and OC the other ray. On OC take any point C. Through C draw any two straight lines. Let OA, OB be cut by one of these in A and B respectively, and by the other in M and L respectively. Join AL and BM and let the joins intersect in N. Join ON. ON is the ray required.

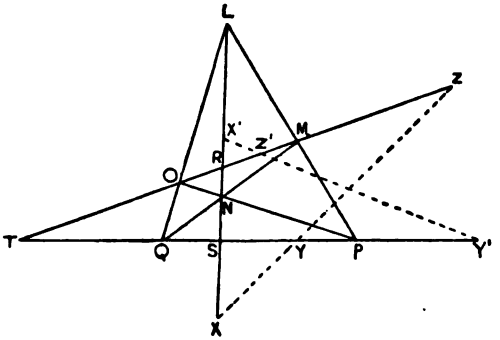
Proof.—The figure ABML is that of a complete quadrangle, of which OA and OB are opposite sides, and OC and ON sides of the diagonal triangle. Therefore O.ABCN is a harmonic pencil (*Art.* 78 *a′*).

The constructions given for *a* and *a′* are plainly reciprocal. The student should, as an exercise, draw a separate figure for *a′*, using the notation for points and lines employed in *Art.* 78 *a′*.

The definitions given (*Arts.* 54, 63) of harmonic ranges and pencils involved considerations of magnitude. Hence harmonic ranges and pencils may be regarded as belonging to the geometry of magnitude. The solutions of the above problems show that they may be equally well regarded as belonging to the geometry of position. It is for this reason that they admit of reciprocation in the sense of *Art* 75.

(82) THEOREM.—*If three points taken on the sides of the*

diagonal triangle of a complete quadrilateral are collinear, their three harmonic conjugates with respect to opposite vertices of the quadrilateral are also collinear.



Proof.—Construct the complete quadrilateral LMNO and its diagonal triangle RST as in the figure of *Art.* 80. Let X, Y, Z be collinear points lying on the sides RS, ST, TR respectively of the diagonal triangle; and let X′, Y′, Z′ be their respective harmonic conjugates with respect to L and N, P and Q, O and M respectively.

Then (*Art.* 61)

$$\frac{RX.RX'}{SX.SX'}=\frac{RL^2}{SL^2};\ \frac{SY.SY'}{TY.TY'}=\frac{SP^2}{TP^2};\ \frac{TZ.TZ'}{RZ.RZ'}=\frac{TM^2}{RM^2}.$$

Therefore, by multiplying,

$$\frac{RX.SY.TZ.RX'.SY'.TZ'}{SX.TY.RZ.SX'.TY'.RZ'}=\frac{RL^2.SP^2.TM^2}{SL^2.TP^2.RM^2}.$$

But (*Art.* 17) since XYZ and LPM are transversals to the triangle RST,

$$\frac{RX.SY.TZ}{SX.TY.RZ}=-1;\ \text{and}\ \frac{RL^2.SP^2.TM^2}{SL^2.TP^2.RM^2}=1.$$

Therefore $\dfrac{RX'.SY'.TZ'}{SX'.TY'.RZ'}=-1.$

Therefore (*Art.* 17 *conv.*) X′, Y′, Z′ are collinear.

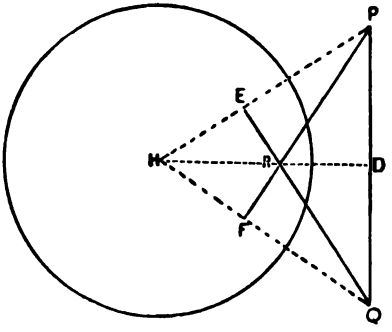
Cor.—The middle points of the diagonals of a complete quadrilateral are collinear.

Proof.—Let X′, Y′, Z′ be the middle points. Then X, Y, Z, their respective harmonic conjugates with respect to opposite vertices, are points at infinity (*Art.* 57, ii.) But all points at infinity are collinear (*Art.* 18). Therefore X, Y, Z are collinear. Therefore X′, Y′, Z′ are collinear.

The following is an easy independent proof of the important theorem of the corollary * :—

Through N, M, P draw parallels to LQ, and through N, O, Q draw parallels to LP, giving two sets of four parallel lines: LAOQ, BNCD, MEFG, PHIK in the one set, and LBMP, ANEH, OCFI, QDGK in the other set. By *Euc.* i. 43, the parallelograms NI, NG are equal, being each equal to LN. ∴ N, F, K are collinear. ∴ the mid-points of LN, LF, LK are collinear. ∴ since the diagonals LF, MO, and the diagonals LK, PQ, bisect each other, the mid-points of LN, MO, PQ are collinear.

(83) THEOREM.—*If* PQR *is a triangle such that* QR *is*



the polar of P, *and* RP *the polar of* Q *with respect to a circle, then* PQ *is the polar of* R *with respect to the circle.*

* Given by Mr J. Dougall in the 'Proceedings of the Edin. Math. Soc.,' vol. xv.

Proof.—The connector of P and Q is the polar of the intersection of QR and RP—*i.e.*, PQ is the polar of R (*Art.* 72).

Def.—A triangle of which each side is the polar of the opposite vertex with respect to a circle, is said to be *self-conjugate* with respect to the circle.

(84) THEOREM.—*One and only one of the vertices of a self-conjugate triangle lies within the circle to which it is self-conjugate.*

Proof.—Let PQR be the triangle. Suppose that all the vertices lie outside the circle. Then the point R and its polar PQ are both external to the circle. But this is impossible (*Art.* 68, ii.) Let R lie within the circle. Then its polar PQ, and consequently the vertices P and Q, lie outside the circle (*Art.* 68, ii.)

Cor.—One and only one of the sides of a self-conjugate triangle lies outside the circle to which it is self-conjugate.

(85) THEOREM.—*If a triangle* PQR *be self-conjugate to a circle whose centre is* H, H *is the orthocentre of the triangle.*

Proof.—Since R is the pole of PQ, and H the centre of the circle, HR is perpendicular to PQ (*Art.* 68). Similarly, HP is perpendicular to QR, and HQ to RP. Therefore H is the orthocentre of the triangle.

Cor.—If HR, HP, HQ meet the opposite sides of the triangle in D, E, F, the radius of the circle $= \sqrt{HR.HD} = \sqrt{HP.HE} = \sqrt{HQ.HF}$.

(86) THEOREM.—*Any triangle which is self-conjugate to a circle is obtuse-angled.*

Proof.—Let the connectors of H with the vertices P and Q, in the figure of *Art.* 83, meet the opposite sides of the triangle PQR in E and F respectively. Then (*Art.* 68), since QE is the polar of P, E lies between P and H. Similarly F lies between Q and H. Therefore QE

and PF lie within the triangle PQH. Therefore their intersection R lies within the triangle PQH. Therefore angle PRQ is greater than angle QEP, which is a right angle.

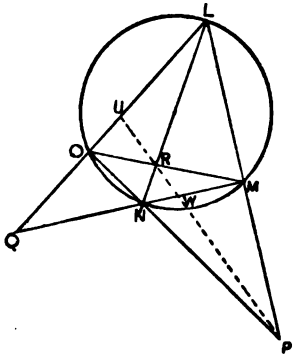
When the triangle is acute-angled, it may be regarded as self-conjugate with respect to an imaginary circle.

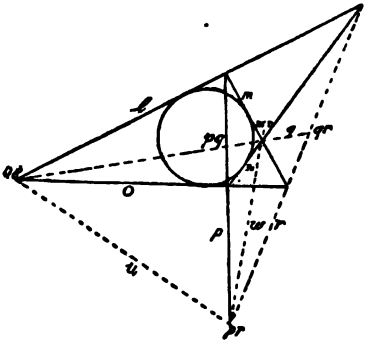
(87) THEOREMS.—

(*a*) *If a complete quadrangle be inscribed in a circle, its diagonal triangle is self-conjugate with respect to the circle.*

(*a'*) *If a complete quadrilateral be circumscribed to a circle, its diagonal triangle is self-conjugate with respect to the circle.*

Proof.—





Let L, M, N, O be the vertices of the inscribed quadrangle, and P, Q, R the vertices of its diagonal triangle. Let W, U be the intersections of PR with MN, OL respectively.

Let *l*, *m*, *n*, *o* be the sides of the circumscribed quadrilateral, and *p*, *q*, *r* the sides of its diagonal triangle. Let *w*, *u* be the connectors of *pr* with *mn*, *ol* respectively.

Then (*Art.* 80, i. *a'*) QWNM and QUOL are harmonic ranges having a common point Q. Therefore WU—*i.e.*, RP—is the polar of Q (*Art.* 68). Similarly PQ is the polar of R, and QR the polar of P.

Then (*Art.* 80, i. *a*) *qwnm* and *quol* are harmonic pencils having a common ray *q*. Therefore *wu*—*i.e.*, *rp*—is the pole of *q* (*Art.* 68). Similarly *pq* is the pole of *r*, and *qr* the pole of *p*.

Cors. i.—

(*a*) If P be regarded as a fixed point, and LM, NO variable chords passing through P, the points Q and R shall lie on a fixed line—viz., the polar of P.

(*a'*) If *p* be regarded as a fixed straight line, and *lm*, *no* variable points lying on *p*, the lines *q* and *r* shall pass through a fixed point—viz., the pole of *p*.

Cors. ii.—

(*a*) The diagonal lines of the complete quadrilateral formed by the tangents at L, M, N, O coincide with the sides of PQR.

(*a'*) The diagonal points of the complete quadrangle formed by the points of contact of *l*, *m*, *n*, *o* coincide with the vertices of *pqr*.

Proof.—

The diagonal line formed by joining the point of intersection of the tangents at L, M, and the point of intersection of the tangents at N, O, is the polar of P (*Art.* 72, *cor.*), and therefore (*Cor.* i. *a*) coincides with QR. Similarly for the other diagonal lines.

The diagonal point formed by the intersection of the connector of the points of contact of *l*, *m*, and the connector of the points of contact of *n*, *o*, is the pole of *p* (*Art.* 73, *cor.*), and therefore (*Cor.* i. *a'*) coincides with *qr*. Similarly for the other diagonal points.

The above two corollaries (*a* and *a'*) may be summed up in the following single statement, which is, obviously, its own reciprocal :—

The diagonal triangle of a complete quadrangle inscribed

in a circle, coincides with the diagonal triangle of the complete quadrilateral whose sides touch the circle at the vertices of the quadrangle.

(88) PROBLEMS.—

(*a*) *To draw, with the ungraduated ruler only, the polar of a given point with respect to a given circle.*

(*a'*) *To find, with the ungraduated ruler only, the pole of a given straight line with respect to a given circle.*

Construction for a (*vide* figure of previous article).—Let P be the given point. Through P draw LM, NO chords to the given circle. Join MN, OL determining the point of intersection Q; and join LN, OM determining the point of intersection R. The connector QR is the polar of P.

Construction for a'.—Take any two points on the given line, and find their polars as in the construction for *a*. The point of intersection of these polars is the pole of the given line.

(89) PROBLEMS (particular cases of last).—

(*a*) *To draw, with the ungraduated ruler only, the tangent at a given point on a circle.*

(*a'*) *To find, with the ungraduated ruler only, the point of contact of a given tangent to a circle.*

Construction.—

Through the given point draw any straight line. Find the pole of the line. The connector of the pole and the given point is the tangent required (*Art.* 69).

On the given tangent take any point. Draw the polar of the point. The point of intersection of the polar and the given tangent is the point of contact required (*Art.* 69).

(90) PROBLEM.—*To draw, with the ungraduated ruler only, two tangents to a circle from a given external point.*

Construction.—Draw the polar of the given point. The

connectors of the given point with the points where the polar cuts the circle are the tangents required (*Art.* 69).

It will be observed that the construction for a' in *Art.* 88, if stated in a form exactly reciprocal to that for a, would involve the drawing of tangents to the circle from external points.

The solutions in *Arts.* 88, 89, and 90 warrant us in regarding poles and polars together with tangents and their points of contact as belonging to the geometry of position. See *Art.* 75, near the end.

CHAPTER VIII.

ANHARMONIC RATIO. PROJECTIVE RANGES AND PENCILS.

(91) *Def.*—If a range consists of four points A, B, C, D, the ratio $\frac{AC}{CB} : \frac{AD}{DB}$ is called the *cross ratio* or *anharmonic ratio* of the range, and is written for brevity (ABCD).

It will be observed that the antecedent, AC/CB, is the position ratio of C with reference to A and B, and that the consequent, AD/DB, is the position ration of D with reference to A and B. In the contraction (ABCD) the order of the letters is important. The two points of reference (regarded as conjugates) come first, the particular point of reference which appears in the numerators of the two position ratios coming first of all. Of the two remaining points (also regarded as conjugates) that one is placed at the end whose position ratio forms the consequent of the cross ratio.

(92) *Particular Cases.*—

(i) If (ABCD) is positive, C and D lie both within, or both without, the finite segment AB.

Proof.—(ABCD) = AC/CB : AD/DB = a positive quantity. Therefore AC/CB and AD/DB are either both positive or both negative. If they are positive, C and D both lie inside the segment AB; if they are negative, C and D both lie outside the segment AB (*Art.* 3).

(ii) If (ABCD) is negative, C and D lie one within and the other without the finite segment AB.

Proof.—(ABCD) = AC/CB : AD/DB = a negative quantity. Therefore, if AC/CB is positive, AD/DB is negative, in which case C lies inside and D outside the segment AB (*Art.* 3). Again, if AC/CB is negative, AD/DB is positive, in which case C lies outside and D inside the segment AB (*Art.* 3).

(iii) If (ABCD) = + 1, C and D coincide.

Proof. — (ABCD) = AC/CB : AD/DB = 1. Therefore AC/CB = AD/DB. Therefore C is the same point as D (*Art.* 4).

(iv) If (ABCD) = − 1, ABCD is a harmonic range.

Proof. — (ABCD) = AC/CB : AD/DB = − 1. Therefore AC/CB = − AD/DB. Therefore ABCD is a harmonic range.

(v) If (ABCD) = 0, either A and C coincide or B and D coincide.

Proof. — (ABCD) = AC/CB : AD/DB = 0. Therefore either AC/CB = 0 or AD/DB = ∞. In the first case A and C coincide; in the second, B and D coincide (*Art.* 3).

(vi) If (ABCD) = ∞, either B and C coincide or A and D coincide.

The proof is similar to that in v.

(93) The letters A, B, C, D can be taken in twenty-four different orders. Thus for a range consisting of the four points A, B, C, D, we have the twenty-four cross ratios enumerated below. These, however, as will be shown, reduce to six *different* ratios. When we speak of *the* cross ratio of a range of four points which are mentioned in a certain order, we mean, as a rule, the particular cross ratio in the contraction for which the letters appear in the same order. Thus (BDCA) is the cross ratio of the range BDCA.

THEOREMS.—*The following relations are true for a range* ABCD :—

(i) (ABCD) = (BADC) = (CDAB) = (DCBA).
(ii) (ABDC) = (BACD) = (CDBA) = (DCAB).
(iii) (ACBD) = (BDAC) = (CADB) = (DBCA).
(iv) (ACDB) = (BDCA) = (CABD) = (DBAC).
(v) (ADBC) = (BCAD) = (CBDA) = (DACB).
(vi) (ADCB) = (BCDA) = (CBAD) = (DABC).

Proof.—(ABCD) = AC/CB : AD/DB = AC.DB/CB.AD = DB/AD : CB/AC = BD/DA : BC/CA = (BADC).

Similarly all the other relations may be proved.

It thus appears that a cross ratio is not affected in value by having any two letters interchanged, if the other two letters are interchanged at the same time.

(94) THEOREM.—*If each of the ratios in group* (i) *above* $= \lambda$, *each of those in* (ii) $= \frac{1}{\lambda}$, *each of those in* (iii) $= 1 - \lambda$, *each of those in* (iv) $= \frac{1}{1-\lambda}$, *each of those in* (v) $= \frac{\lambda - 1}{\lambda}$, *and each of those in* (vi) $= \frac{\lambda}{\lambda - 1}$.

Proof.—

$$(ABDC) = AD/DB : AC/CB = \frac{1}{AC/CB : AD/DB} = \frac{1}{(ABCD)} = \frac{1}{\lambda}. \quad (a)$$

Again, AB.CD + BC.AD + CA.BD = 0 (*Art.* 10).

$$\therefore \frac{AB.DC}{BC.AD} = \frac{BC.AD}{BC.AD} - \frac{AC.DB}{CB.AD}.$$

$$\therefore AB/BC : AD/DC = 1 - AC/CB : AD/DB.$$

$$\therefore (ACBD) = 1 - (ABCD) = 1 - \lambda. \quad (b)$$

It may be proved, as in (*a*) that

$$(ACDB) = \frac{1}{(ACBD)} = \frac{1}{1-\lambda},$$

as in (*b*) that $(ADBC) = 1 - (ABDC) = 1 - \frac{1}{\lambda} = \frac{\lambda - 1}{\lambda}$, and as in (*a*) that $(ADCB) = \frac{1}{(ADBC)} = \frac{\lambda}{\lambda - 1}$.

From this theorem it appears that the range ABCD has, in general, six *different* cross ratios, three of which are the reciprocals of the other three.

Cor.—If a cross ratio of one range of four points is equal to a cross ratio of another range of four points, the two ranges have all their cross ratios respectively equal.

Def.—Ranges of four points which have their cross ratios respectively equal, are said to be *equicross* or *equianharmonic.*

(95) THEOREM.—*If a range* ABCD, *no two of whose points are coincident, is such that the ratio* (ABCD) *is unaffected in value by having any two letters interchanged when the other two are not interchanged at the same time, the points indicated by the interchanged letters are harmonic conjugates with respect to the other two.*

Proof.—By making the interchanges indicated in all possible ways, we obtain from (ABCD) the following six ratios:—(ABDC), (BACD), (ACBD), (DBCA), (ADCB), (CBAD). But of these the first two, the middle two, and the last two are equal (*Art.* 93). We have thus only three cases to deal with.

(i) Let (ABCD) = (ABDC). Then $\lambda = \frac{1}{\lambda}$. $\therefore \lambda = \pm 1$.

$\lambda = +1$ is inadmissible, as no two points on the range are coincident (*Art.* 92, iii). $\therefore \lambda = -1$. $\therefore$ (ABCD) $= -1$. Therefore (*Art.* 92, iv) ABCD is a harmonic range, C and D (the interchanged letters) being harmonic conjugates with respect to A and B.

(ii) Let (ABCD) = (ACBD).

Then $\lambda = 1 - \lambda$. $\therefore \frac{\lambda - 1}{\lambda} = -1$. $\therefore$ (ADBC) $= -1$.

Therefore ADBC is a harmonic range, B and C (the interchanged letters) being harmonic conjugates with respect to A and D.

(iii) Let (ABCD) = (ADCB). Then $\lambda = \frac{\lambda}{\lambda - 1}$. $\therefore$ either $\lambda = 0$ or $\lambda = 2$. But $\lambda = 0$ is inadmissible, as no two points on the range are coincident (*Art.* 92, v). $\therefore \lambda = 2$. $\therefore \frac{1}{1-\lambda} = -1$. $\therefore$ (ACDB) $= -1$. Therefore ACDB is a harmonic range, D and B (the interchanged letters) being harmonic conjugates with respect to A and C.

(96) *Def.*—

If (ABCD) = (A′B′C′D′), the pairs of points A and A′, B and B′, C and C′, D and D′, which occupy similar positions in the contracted expressions for the ratios, are said to be pairs of *corresponding* points.

If (ABCD) = (A′B′C′D′), (ABCD) is also equal to (B′A′D′C′), to (C′D′A′B′), and to (D′C′B′A′) (*Art.* 93). It follows that the points of two equianharmonic ranges ABCD, A′B′C′D′, can, in general, be made to correspond to each other in pairs, in four different ways. In the particular case when the ranges are harmonic, their points can, as we have seen (*Art.* 76), be made to correspond in eight different ways.

(97) THEOREM.—*If two ranges,* ABCDEFG . . . *and* A′B′C′D′E′F′G′ . . ., *whose points are regarded as corresponding in pairs,* A *and* A′, B *and* B′, C *and* C′, *&c., are such that* (ABCD) = (A′B′C′D′), (ABCE) = (A′B′C′E′), (ABCF) = (A′B′C′F′), *&c., then every four points on the one range and the corresponding four points on the other form a pair of equianharmonic ranges.*

Proof.—

(ABCD) = (A′B′C′D′).

$\therefore$ AC/CB : AD/DB = A′C′/C′B′ : A′D′/D′B′.

Also (ABCE) = (A′B′C′E′).

$\therefore$ AC/CB : AE/EB = A′C′/C′B′ : A′E′/E′B′.

$\therefore$ AD/DB : AE/EB = A′D′/D′B′ : A′E′/E′B′.

$\therefore$ (ABDE) = (A′B′D′E′). (1)

Similarly (ABDF) = (A'B'D'F').

From the last two results we obtain (BDAE) = (B'D'A'E') and (BDAF) = (B'D'A'F').

Hence it may be proved as for (1) that (BDEF) = (B'D'E'F'). (2)

Similarly (BDEG) = (B'D'E'G').

From the last two results we obtain (DEBF) = (D'E'B'F') and (DEBG) = (D'E'B'G').

Hence it may be proved as for (1) that (DEFG) = (D'E'F'G'). (3)

Result (1) involves *two* of the points A, B, C which appear in all the given ratios referring to the first range, result (2) involves *one* of these points, and result (3) involves *none* of them. The above three cases are therefore typical of all the others, which may be similarly established.

Def.—Two ranges, ABCDEF . . . and A'B'C'D'E'F' . . ., whose points are regarded as corresponding in pairs, A and A', B and B', C and C', &c., are said to be *homographic* when every four points of the one and the corresponding four points of the other form a pair of equianharmonic ranges.

The expression (ABC . . .) = (A'B'C' . . .) may be used as a contraction for the statement that the two ranges ABC . . ., A'B'C' . . ., are homographic.

(98) *Defs.*—

(i) (α) A range consisting of the points of intersection of a transversal with the rays of a pencil, is said to be a *section* of the pencil.

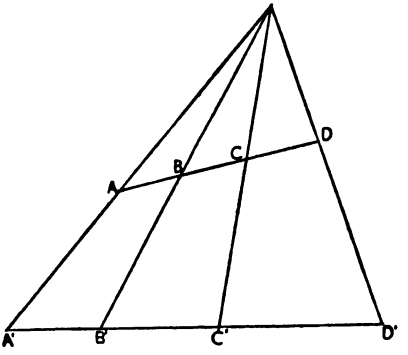
(α') A pencil consisting of the connectors of a point with the points of a range, is said to be a *projection* of the range.

The following notation is often useful:—*s.abc* . . . represents the range consisting of the points of intersection of a straight line *s* and the rays, *abc* . . . of a pencil, *abc* . . . ; *s*(*abcd*) represents the anharmonic ratio of the range *s.abcd*; *s*(*abc* . . .) = *s'*(*a'b'c* . . .) indicates that the ranges *s.abc* . . ., *s'.a'b'c'* . . ., are homographic.

(ii) (α) Ranges are said to be *in perspective*, if they are sections of the same pencil. The vertex of the pencil is called the *centre of perspective* of the ranges.

(α') Pencils are said to be *in perspective*, if they are projections of the same range. The axis of the range is called the *axis of perspective* of the pencils.

(99) THEOREM.—*Ranges in perspective are homographic.*



Proof.—Let the ranges ABC . . ., A′B′C′ . . . be sections of the same pencil, and let A, B, C, D, be any four points of the first, and A′, B′, C′, D′, the four points of the second which lie on the same rays of the pencil. Then it can be proved exactly as in *Art.* 62 that

$$\frac{AC}{CB} \cdot \frac{A'D'}{B'D'} = \frac{A'C'}{C'B'} \cdot \frac{AD}{BD}.$$

$$\therefore \frac{AC}{CB} : \frac{AD}{BD} = \frac{A'C'}{C'B'} : \frac{A'D'}{B'D'}.$$

i.e. (ABCD) = (A′B′C′D′).

Similarly it can be proved that any other four points of ABC . . ., and the four points of A′B′C′ . . . which lie on the same rays of the pencil, form two homographic ranges.

Therefore (*Art.* 97, *def.*) (ABC . . .) = (A′B′C′ . . .).

Cor.—All sections of a pencil of four rays are equianharmonic.

(100) *Defs.*—

(i) A pencil of four rays is said to have the same anharmonic ratios as any of its sections. The anharmonic ratio of the pencil S.ABCD, the order of the rays being taken into account, is written for brevity S(ABCD), that of the pencil *abcd* is written (*abcd*).

It is evident that theorems similar to those given in *Arts.* 92, 93, 94, 95, and 97 for ranges, hold also for pencils.

(ii) Pencils of four rays which have their cross ratios respectively equal are said to be *equicross* or *equianharmonic.*

Corresponding rays of equianharmonic pencils may be defined in the same way as corresponding points of equianharmonic ranges.

(iii) Two pencils, S.ABCDEF... and S'.A'B'C'D'E'F'..., whose rays are regarded as corresponding in pairs SA and S'A', SB and S'B', SC and S'C', &c., are said to be *homographic* when every four rays of the one and the corresponding four rays of the other form a pair of equianharmonic pencils.

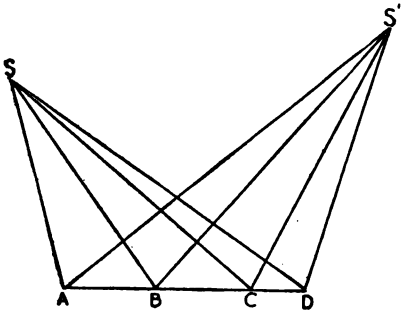
The expression $S(ABC\ldots)=S'(A'B'C'\ldots)$ may be used as a contraction for the statement that the two pencils S.ABC..., S'.A'B'C'... are homographic. If the pencils are represented by *abc*..., *a'b'c'*..., the contraction used is $(abc\ldots)=(a'b'c'\ldots)$.

(101) THEOREM (reciprocal of *Art.* 99).—*Pencils in perspective are homographic.*

Proof.—Let the pencils S.ABC..., S'.ABC... be projections of the same range, and let SA, SB, SC, SD be any four rays of the first, and S'A, S'B, S'C, S'D the four rays of the second which pass through the same points, A, B, C, D, of the range.

Then S(ABCD) = (ABCD) = S′(ABCD) (*Art.* 100, i).

Similarly it can be proved that any other four rays of

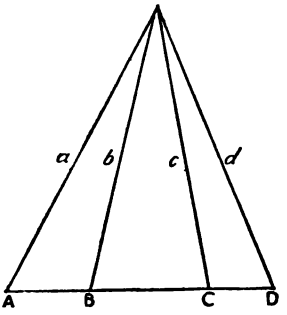


S.ABC . . . and the four rays of S′.ABC . . . which pass through the same points of the range, form two equianharmonic pencils.

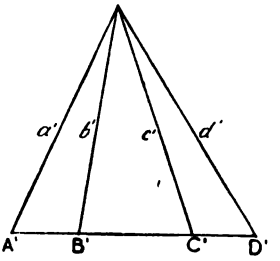
Therefore (*Art.* 100, iii) S(ABC . . .) = S′(ABC . . .).

Cor.—All projections of a range of four points are equianharmonic.

(102) THEOREMS.—



(α) *Sections of homographic (or equianharmonic) pencils are homographic (or equianharmonic).*



(α′) *Projections of homographic (or equianharmonic) ranges are homographic (or equianharmonic).*

Proof.—

Let a, b, c, d be any four rays of a pencil, a', b', c', d', the corresponding rays of a homographic pencil. Let a section of the first pencil cut a, b, c, d in A, B, C, D respectively, and let a section of the second cut a', b', c', d' in A′, B′, C′, D′ respectively. Then, by *Art.* 100, i. (ABCD) = $(abcd)$, and (A′B′C′D′) = $(a'b'c'd')$. But $(abcd) = (a'b'c'd')$. Therefore (ABCD) = (A′B′C′D′). Similarly any other four points of one section are equianharmonic with four points of the other. Therefore the sections are homographic.

Let A, B, C, D be any four points of a range, A′, B′, C′, D′, the corresponding points of a homographic range. Let a projection of the first range have the rays a, b, c, d passing through A, B, C, D respectively, and let a projection of the second have the rays a', b', c', d' passing through A′, B′, C′, D′ respectively. Then, by *Art.* 100, i., $(abcd)$ = (ABCD), and $(a'b'c'd')$ = (A′B′C′D′). But (ABCD) = (A′B′C′D′). Therefore $(abcd) = (a'b'c'd')$. Similarly any other four rays of one projection are equianharmonic with four rays of the other. Therefore the projections are homographic.

(103) *Defs.*—

(α) If a point of one of two homographic ranges coincides with the corresponding point of the other range, the point is said to be *self-corresponding.*

(α′) If a ray of one of two homographic pencils coincides with the corresponding ray of the other pencil, the ray is said to be *self-corresponding.*

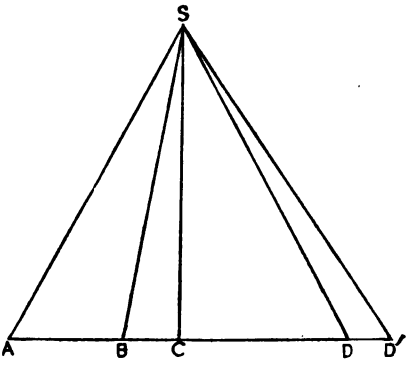
THEOREMS.—

(α) *Two homographic ranges having three self-corresponding points, have all their points self-corresponding.*

(α′) *Two homographic pencils having three self-corresponding rays, have all their rays self-corresponding.*

Proof of α.—Let A, B, C be the three self-corresponding points, D any other point on the first range, and D′ the

corresponding point on the second. Then (ABCD) = (ABCD')—*i.e.*, $\frac{AC}{CB}/\frac{AD}{DB} = \frac{AC}{CB}/\frac{AD'}{D'B}$. $\therefore \frac{AD}{DB} = \frac{AD'}{D'B}$.



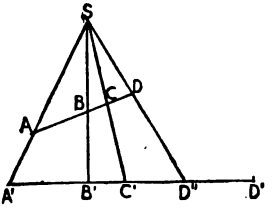
Therefore D and D' coincide.

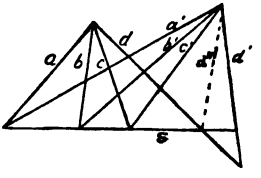
Proof of α'.—Let SA, SB, SC be the three self-corresponding rays, SD any other ray of the first pencil, SD' the corresponding ray of the second. Let a transversal meet the five rays in A, B, C, D, and D' respectively. Then S(ABCD) = S(ABCD'). ∴ (ABCD) = (ABCD'). ∴ D and D' coincide. ∴ SD and SD' coincide.

(104) THEOREMS.—

(α) *If three points of one of two homographic ranges connect concurrently with the three corresponding points of the other range, the ranges are in perspective.*

(α') *If three rays of one of two homographic pencils intersect collinearly with the three corresponding rays of the other pencil, the pencils are in perspective.*





Proof.—

Let ABC . . ., A'B'C' . . . be the two ranges, and let the

Let *abc* . . ., *a'b'c'* . . . be the two pencils, and let the points

connectors AA', BB', CC' pass through the same point S. Let D be any point of ABC . . ., and D' the corresponding point of A'B'C'. . . . Let the connector SD meet the axis of A'B'C' . . . in D''.

Then, by *Art.* 99, *Cor.* (ABCD) = (A'B'C'D''). But (A B C D) = (A' B' C' D'). Therefore (A' B' C' D') = (A'B'C'D''). Therefore, by *Art.* 103, D' coincides with D''. Therefore DD' passes through S.

Similarly it could be proved that the connector of any other pair of corresponding points passes through S. Therefore the ranges are in perspective.

of intersection aa', bb', cc' lie on the same straight line s. Let d be any ray of abc . . ., and d' the corresponding ray of $a'b'c'$. . . . Let the point of intersection sd connect with the vertex of $a'b'c'$ . . . by d'.

Then, by *Art.* 101, *Cor.* $(a\,b\,c\,d) = (a'\,b'\,c'\,d'')$. But $(abcd) = (a'b'c'd')$. Therefore $(a'b'c'd') = (a'b'c'd'')$. Therefore, by *Art.* 103, d' coincides with d''. Therefore dd' lies on s.

Similarly it could be proved that the point of intersection of any other pair of corresponding rays lies on s. Therefore the pencils are in perspective.

Cor.—The theorems are true for equianharmonic ranges and pencils.

The theorems of *Art.* 77 are particular cases of the above.

(105) THEOREMS.—

(α) *Two homographic ranges which have one self-corresponding point, and whose axes do not coincide, are in perspective.*

Proof.—

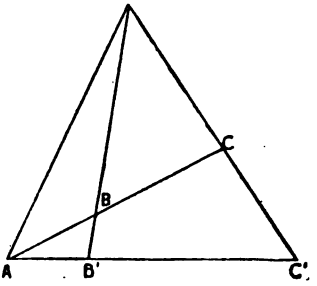
Let the ranges be ABC . . ., AB'C' . . ., A being the self-

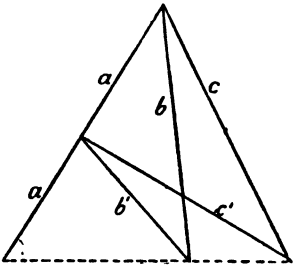
(α') *Two homographic pencils which have one self-corresponding ray, and whose vertices do not coincide, are in perspective.*

Let the pencils be abc . . ., $ab'c'$ . . ., a being the self-

corresponding point. Then any straight line through A can be regarded as the connector of the point A of one range and the corresponding point A of the other. Therefore the connectors of A and A, B and B', C and C' are concurrent. Therefore (*Art.* 104) the ranges are in perspective.

corresponding ray. Then any point on *a* can be regarded as the point of intersection of the ray *a* of one pencil and the corresponding ray *a* of the other. Therefore the points of intersection of *a* and *a*, *b* and *b'*, *c* and *c'* are collinear. Therefore (*Art.* 104) the pencils are in perspective.





Cor.—The theorems are true for equianharmonic ranges and pencils.

(106) *Def.*—Two ranges, or two pencils, are said to be *projective* when one can be derived from the other by a finite number of projections and sections.

Thus in the figure of α below, ABCD, A'B'C'D' are projective ranges, the intermediate operations by means of which one is derived from the other being projection S.ABCD, section A''B''C''D'', projection S'.A''B''C''D''. In the figure of α', *abcd*, *a'b'c'd'* are projective pencils, the one being derived from the other by the following intermediate operations, section *s.abcd*, projection *a''b''c''d''*, section *s'.a''b''c''d''*.

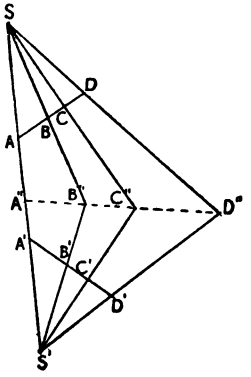
THEOREMS.—*Two projective* { *ranges* / *pencils* } *are homographic.*

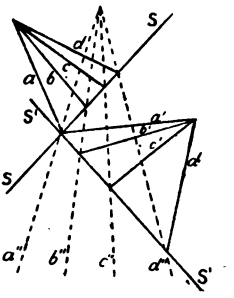
Proof.—The two projective { ranges / pencils } form the first and the last in a series of { ranges / pencils } each of which is in perspective with the next in order (*Def.*). Therefore each { range / pencil } in the series is homographic with the { range / pencil } next in order (*Arts.* 99 and 101). Therefore all the { ranges / pencils } in the series are homographic.

Converses.—

(*a*) Two homographic ranges which are not in perspective are projective.

(*a'*) Two homographic pencils which are not in perspective are projective.





Proof.—

Let S and S' be any two points on the connector of the corresponding points A and A' of the homographic ranges ABCD . . ., A'B'C'D' . . .,

Let *s* and *s'* be any two straight lines through the point of intersection of the corresponding rays *a* and *a'* of the homographic pencils

and let S.ABCD . . ., S'.A'B'C'D' . . . be projections of these ranges. Then S.ABCD . . ., S'.A'B'C'D' . . . are homographic (*Art.* 102, α'). They have also a self-corresponding ray SS'. Therefore (*Art.* 105, α') they are in perspective. Therefore ABCD . . ., A'B'C'D' . . . are projective.

abcd . . ., *a'b'c'd'* . . ., and let *s.abcd* . . ., *s'.a'b'c'd'* . . . be sections of these pencils. Then *s.abcd* . . ., *s'.a'b'c'd'* . . . are homographic (*Art.* 102, α). They have also a self-corresponding point *ss'*. Therefore (*Art.* 105, α) they are in perspective. Therefore *abcd* . . ., *a'b'c'd'* . . . are projective.

The above constructions evidently furnish solutions of the following problems :—

Two homographic or projective $\left\{\begin{matrix}\text{ranges}\\\text{pencils}\end{matrix}\right\}$ being given, to construct a third $\left\{\begin{matrix}\text{range}\\\text{pencil}\end{matrix}\right\}$ with which each shall be in perspective.

The points S and S' can be taken on any of the other connectors of corresponding points, such as BB', CC', &c. They can also be taken anywhere on the selected connector, and they may coincide with each other. Similar remarks apply to the lines *s* and *s'*. The above problems, therefore, admit of an indefinite number of solutions. Interesting particular cases which will be dealt with in *Art.* 107, arise by making $\left\{\begin{matrix}\text{S}\\s\end{matrix}\right\}$ and $\left\{\begin{matrix}\text{S}'\\s'\end{matrix}\right\}$ coincide respectively with $\left\{\begin{matrix}\text{A}'\\a'\end{matrix}\right\}$ and $\left\{\begin{matrix}\text{A},\\a,\end{matrix}\right\}$ then with $\left\{\begin{matrix}\text{B}'\\b'\end{matrix}\right\}$ and $\left\{\begin{matrix}\text{B},\\b,\end{matrix}\right\}$ &c.

Since homographic ranges and pencils are either projective or in perspective, they are capable of being constructed by the ungraduated ruler only, and hence admit of reciprocation in the sense of *Art.* 75.

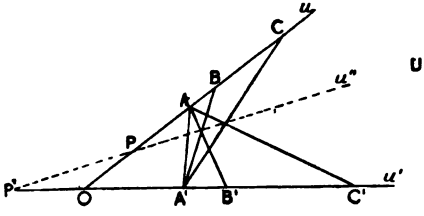
(107) THEOREMS.—

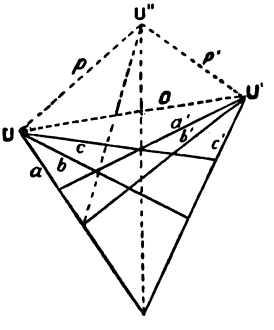
(α) *If* ABC . . ., A′B′C′ . . . *are two projective ranges on different axes*, u *and* u′, *the following pairs of pencils are in perspective :—*

A.A′B′C′ . . ., A′.ABC . . . ;
B.A′B′C′ . . ., B′.ABC . . . ;
C.A′B′C′ . . ., C′.ABC . . . ;
&c.

(α′) *If* abc . . ., a′b′c′ . . . *are two projective pencils with different vertices*, U *and* U′, *the following pairs of ranges are in perspective :—*

a.a′b′c′ . . ., a′.abc . . . ;
b.a′b′c′ . . ., b′.abc . . . ;
c.a′b′c′ . . ., c′.abc . . . ;
&c.





Proof.—The { pencils / ranges } forming any one of the pairs referred to are homographic, and have a self-corresponding { ray. / point. } Therefore they are in perspective.

Cors. i.—

(α) Let u'' be the line in which the corresponding rays of the pencils A.A′B′C′ . . ., A′.ABC . . . intersect, and let u'' cut u in P and u' in P′, and let O be the point of intersection of u and u'.

(α′) Let U″ be the point in which the connectors of corresponding points of the ranges *a.a′b′c′* . . ., *a′.abc* . . . intersect, and let p be the connector of U″ and U, p' the connector of U″ and U′, and o the connector of U and U′.

Then P in range ABC . . . corresponds to O in range A′B′C′ . . ., and P′ in range A′B′C′ . . . to O in range ABC . . .

Then p in pencil abc corresponds to o in pencil $a'b'c'$ . . ., and p' in pencil $a'b'c'$ . . . to o in pencil abc . . .

Proof.—

PABC . . . is a section of A′.PABC . . .; and OA′B′C′ . . . is a section of A.OA′B′C′ . . . But, since A′P and AO intersect on u'', A′.PABC . . ., A.OA′B′C′ . . . are in perspective. Therefore (PABC . . .) = (OA′B′C . . .).

Similarly P′ in range A′B′C′ . . . may be proved to correspond to O in range ABC . . .

$pabc$ . . . is a projection of $a'.pabc$ . . .; and $oa'b'c'$ . . . is a projection of $a.oa'b'c'$ . . . . But, since $a'p$ and ao connect through U″, $a'.pabc$ . . ., $a.oa'b'c'$ . . . are in perspective. Therefore $(pabc\ .\ .\ .) = (oa'b'c'\ .\ .\ .)$.

Similarly p' in pencil $a'b'c'$ . . . may be proved to correspond to o in pencil abc . . .

Cors. ii.—

(α) The corresponding rays of each of the pairs of pencils mentioned in the theorem intersect on the same line u''.

(α′) The corresponding points of each of the pairs of ranges mentioned in the theorem connect through the same point U″.

Proof.—

Let the axis of perspective of any pair cut u in Q and u' in Q′. Then it may be proved as in *Cor.* i. α, that Q in ABC . . . corresponds to O in A′B′C′ . . ., and that Q′ in A′B′C′ . . . corresponds to O in ABC . . . . Therefore Q coincides with P, and Q′ with P′. Therefore the axis of perspective of each

Let the centre of perspective of any pair connect with U by means of q, and with U′ by means of q'. Then it may be proved as in *Cor.* i. α′, that q in abc . . . corresponds to o in $a'b'c'$ . . ., and that q' in $a'b'c'$ . . . corresponds to o in abc . . . . Therefore q coincides with p, and q' with p'. There-

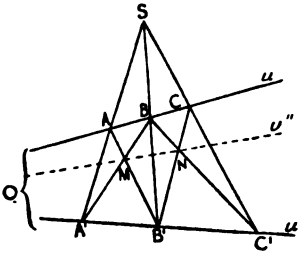
pair of pencils mentioned passes through P and P′, and hence coincides with u''.

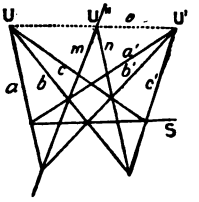
fore the centre of perspective of each pair of ranges mentioned lies on p and p', and hence coincides with U″.

(108) PROBLEMS.—

(α) *To draw, with the ungraduated ruler only, the straight line connecting a given point* M *with the point of intersection* O, *regarded as inaccessible, of two given straight lines* u *and* u′.

(α′) *To find, with the ungraduated ruler only, the point of intersection of a given straight line* m *with the connector* o, *regarded as incapable of being drawn, of two given points* U *and* U′.





Construction.—

Through M draw two straight lines cutting u in the points A and B, and u' in the points B′ and A′ respectively. Let S be the point of intersection of the connectors AA′, BB′. Through S draw any straight line cutting u in C, and u' in C′. Let N be the point of intersection of the connectors BC′, B′C.

In m take two points connecting with U by the straight lines a and b, and with U′ by the straight lines b' and a' respectively. Let s be the connector of the points of intersection aa', bb'. On s take any point connecting with U by the straight line c, and with U′ by the straight line c'. Let n be the con-

The connector of M and N is the line, u'', required.

nector of the points of intersection bc', $b'c$. The point of intersection of m and n is the point, U'', required.

Proof.—

The ranges OABC, OA'B'C' are in perspective. Therefore the pencils A.OA'B'C', A'.OABC are homographic, and they have a self-corresponding ray. Therefore they are in perspective. Therefore O, M, N, are collinear.

The pencils *oabc*, *oa'b'c'* are in perspective. Therefore the ranges *a.oa'b'c'*, *a'.oabc* are homographic, and they have a self-corresponding point. Therefore they are in perspective. Therefore *o*, *m*, *n*, are concurrent.

(109) THEOREMS.—

(α) *Any two ranges* ABC, A'B'C', *each consisting of three points, are projective.*

(α') *Any two pencils* abc, a'b'c', *each consisting of three rays, are projective.*

Proof.—

Let S, S' be any two points on the connector AA'. Let SB, S'B' intersect in B'', and SC, S'C' in C'', and AA', B''C'' in A''. Then ABC, A'B'C' are each in perspective with A''B''C''. Therefore they are projective.

Let *s*, *s'* be any two straight lines through the point of intersection *aa'*. Let *b''* be the connector of *sb*, *s'b'*, and *c''* the connector of *sc*, *s'c'*, and *a''* the connector of *aa'*, *b''c''*. Then *abc*, *a'b'c'* are each in perspective with *a''b''c''*. Therefore they are projective.

Unless it is otherwise stated, projective ranges and pencils are to be regarded as having at least four points or rays.

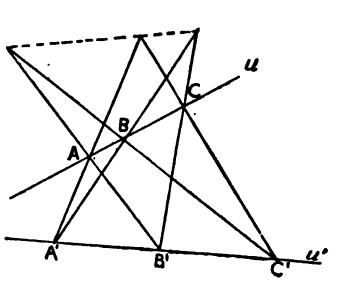
(110) THEOREMS.—

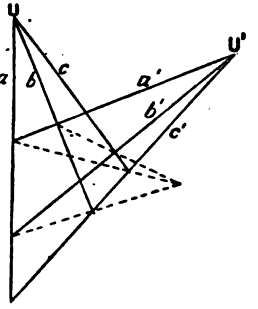
(α) *If any three points be taken on a straight line* u,

(α') *If any three straight lines be taken passing through a*

and any other three points on another straight line u′, *and if these six points be joined in cyclic order by six connectors, so that in the order adopted, points on* u *alternate with points on* u′, *then the points of intersection of the following pairs of connectors are collinear:*—1*st and* 4*th; 2nd and* 5*th; 3rd and* 6*th.*

point U, *and any other three straight lines passing through another point* U′, *and if these six lines intersect in cyclic order in six points of intersection, so that in the order adopted, lines through* U *alternate with lines through* U′, *then the connectors of the following pairs of points of intersection are concurrent:*—1*st and* 4*th; 2nd and* 5*th; 3rd and* 6*th.*





Proof.—

Let A, B, C be the three points on *u*, and A′, B′, C′ the three points on *u′*. Join the six points in, say, the following order:—

AA′BC′CB′—*i.e.* draw in order the following connectors:—AA′, A′B, BC′, C′C, CB′, B′A.

Let *a*, *b*, *c* be the three lines through U, and *a′ b′ c′* the three lines through U′. Let the six lines intersect in pairs in, say, the following order:—

aa′bc′cb′—*i.e.* take in order the following points of intersection:—*aa′*, *a′b*, *bc′ c′c*, *cb′*, *b′a*.

Then (*Art.* 109) ABC, C′B′A′ are projective ranges. Therefore the following pairs of pencils are in perspective (*Art.* 107, α), viz.—

A.C′B′A′, C′.ABC;

B.C′B′A′, B′.ABC.

Therefore the following pairs of lines intersect on the same straight line, viz., AB′ and C′B, AA′ and C′C, BA′ and B′C (*Art.* 107, ii).

The six points may be joined in the manner indicated in six different orders, each giving rise to a distinct figure. The orders are:—

AA′BB′CC′, AB′CA′BC′,
AA′CB′BC′, AA′BC′CB′,
AA′CC′BB′, AB′BA′CC′.

Then (*Art.* 109) *abc*, *c′b′a′* are projective pencils. Therefore the following pairs of ranges are in perspective (*Art.* 107, α′), viz.—

a.c′b′a′, *c′.abc*;

b.c′b′a′, *b′.abc*.

Therefore the following pairs of points connect through the same point, viz., *ab′* and *c′b*, *aa′* and *c′c*, *ba′* and *b′c* (*Art.* 107, ii).

The six lines intersect, in the manner indicated in six different orders, each giving rise to a distinct figure. The orders are:—

aa′bb′cc′, *ab′ca′bc′*,
ac′cb′bc′, *aa′bc′cb′*,
aa′cc′bb′, *ab′ba′cc′*.

The proofs are similar for all the cases. The student will find it instructive to draw the figures for himself. The present theorems are particular cases of those of Pascal and Brianchon (*Art.* 113).

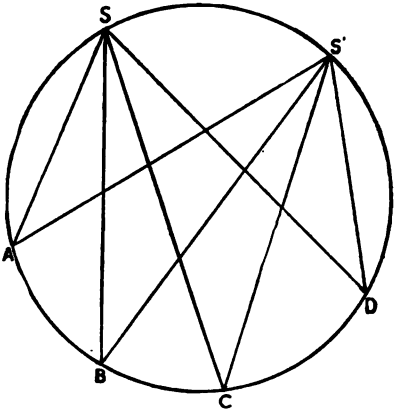
(111) THEOREMS.—

(α) *The connectors of any two points on a circle with any series of points on the circle, form two projective pencils.*

(α′) *The points of intersection of any two tangents to a circle with any series of tangents to the circle, form two projective ranges.*

Proof of α.—Let S and S′ be two points on a circle, and ABCD . . . any series of points on the circle. Join S and S′ with A, B, C, D . . . . Then the following pairs of angles are either equal or supplementary, viz., ASB,

AS'B; BSC, BS'C; CSD, CS'D, &c. Therefore the two pencils S.ABCD . . ., S'.ABCD . . . may be superposed.

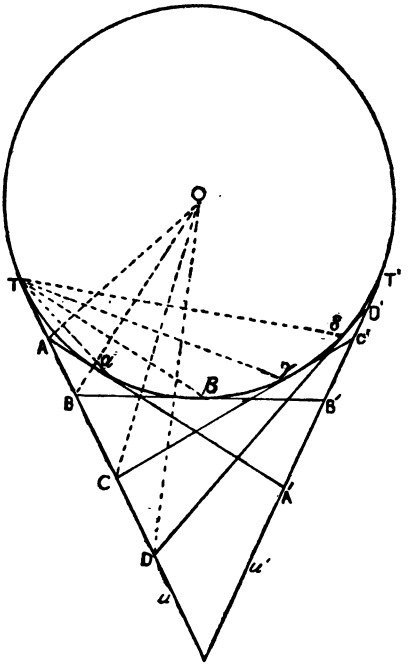


Therefore the pencils are homographic, and, therefore, projective.

Proof of α'.—Let u and u' be two tangents touching a circle at T and T' respectively. Let α, β, γ, δ . . . be any series of points on the circle. Draw tangents at α, β, γ, δ . . . cutting u in A, B, C, D . . ., and u' in A', B', C', D' . . . respectively. Connect T and T' with α, β, γ, δ . . ., and O, the centre of the circle, with A, B, C, D . . ., and with A', B', C', D' . . . .

Then it may easily be proved that OA, OB, OC, OD . . . are perpendicular to Tα, Tβ, Tγ, Tδ . . . . Therefore the following pairs of angles are either equal or supplementary, viz., AOB, αTβ; BOC, βTγ; COD, γTδ, &c. Therefore the pencils O.ABCD . . ., T.$\alpha\beta\gamma\delta$ . . . are superposable. Similarly it may be proved that the pencils O.A'B'C'D' . . ., T'.$\alpha\beta\gamma\delta$ . . . are superposable. But T.$\alpha\beta\gamma\delta$ . . ., T'.$\alpha\beta\gamma\delta$ . . . are homographic. Therefore O.ABCD . . ., O.A'B'C'D' . . . are homographic. Therefore their sections ABCD . . ., A'B'C'D' . . . are homographic, and therefore projective.

The proofs of α and α' are an instance of reciprocation where the magnitude of angles is taken into account.

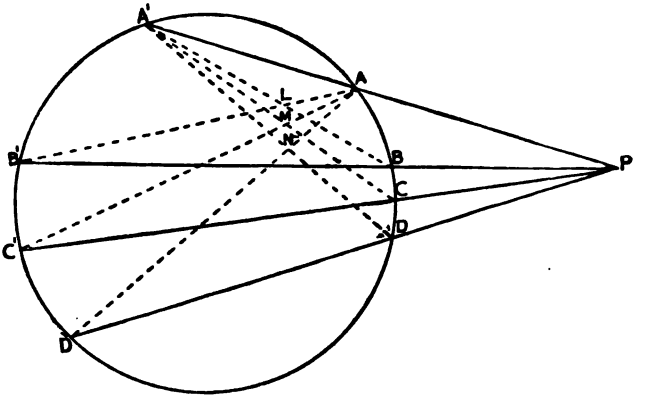


(112) *Def.*—If SABC . . ., S'A'B C' . . . are two series of points on the same circle, or on two different circles, such that the pencils S.ABC . . ., S'.A'B'C' . . . are homographic, the series ABC . . ., A'B'C' . . . are said to be homographic. This is briefly expressed thus: (ABC . . .) = (A'B'C' . . .).

THEOREM.—*If through* P, *any point inside or outside a circle, chords* AA', BB', CC' DD', *&c., are drawn to the circle,* (ABCD . . .) = (A'B'C'D' . . .).

Proof.—Let the connectors AB', A'B intersect in L;

AC', A'C in M; AD' A'D in N, &c. Then L, M, N, &c., lie on the polar of P, and are therefore collinear.



Hence A.A'B'C'D' . . ., A'.ABCD . . . are in perspective. Therefore (ABCD . . .) = (A'B'C'D . . .).

(113) *Def.*—If six { points / straight lines } be taken in any cyclic order, and the { connector / point of intersection } of each with the next in order (including the { connector / point of intersection } of the last with the first) be taken into account, the resulting figure, consisting of six { vertices / sides } and six { sides / vertices } is called a *simple* (as distinguished from a *complete*) { *hexagon* / *six-side* or *hexagon* }

The notation used indicates the order in which the points or lines are to be taken. Thus AECBFD denotes the simple hexagon formed by the six points A, E, C, B, F, D and the six connectors AE, EC, CB, BF, FD, DA. It is plain that AECBFD and FDAECB, in which the same

order of letters is maintained, are the same hexagon. AECBFD also plainly denotes the same hexagon as DFBCEA, in which the order is reversed but the same juxtaposition of letters preserved. AECBFD and ABCDEF, on the other hand, are evidently different hexagons. The total number of ways in which six letters can be arranged in cyclic order (juxtaposition only being taken into account) is $\frac{\lfloor 5}{2}$ or 60. Thus with six points or six lines 60 simple hexagons can be formed.

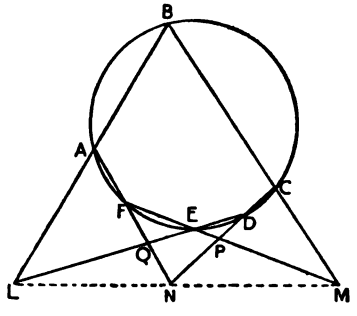
The following pairs of vertices or sides are said to be *opposite*, viz., the 1st and 4th, the 2nd and 5th, the 3rd and 6th, in the order indicated by the notation used for any particular hexagon. Thus in the hexagon ABCDEF, the following pairs of vertices are opposite, viz., A and D, B and E, C and F; and the following pairs of sides are opposite, viz., AB and DE, BC and EF, CD and FA.

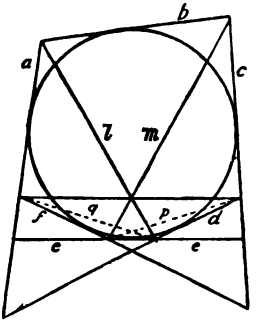
(α) PASCAL'S THEOREM.

The points of intersection of the opposite sides of a simple hexagon inscribed in a circle are collinear.

(α') BRIANCHON'S THEOREM.

The connectors of the opposite vertices of a simple six-side (hexagon) circumscribed to a circle are concurrent.





Proof.—

Let ABCDEF be a simple hexagon inscribed in a circle.

Let *abcdef* be a simple hexagon circumscribed to

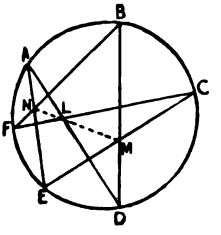
Let the opposite sides AB and DE, BC and EF, CD and FA intersect in L, M, and N respectively. Let CD, EF intersect in P, and DE, FA in Q.

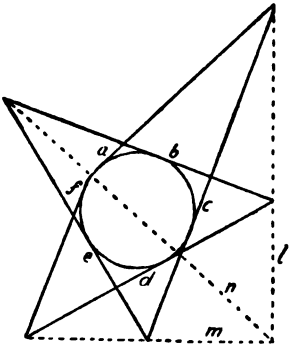
Then (*Art.* 111, α) the pencils A.BDEF and C.BDEF are projective. Therefore their sections LDEQ and MPEF are projective, and they have a self-corresponding point E. Therefore they are in perspective. Therefore LM, DP, QF, the connectors of corresponding points, are concurrent, *i.e.* L, M, N are collinear.

a circle. Let the opposite vertices *ab* and *de*, *bc* and *ef*, *cd* and *fa* connect by *l*, *m*, and *n* respectively. Let *cd*, *ef* connect by *p*, and *de*, *fa* by *q*.

Then (*Art.* 111, α′) the ranges *a.bdef* and *c.bdef* are projective. Therefore their projections *ldeq* and *mpef* are projective, and they have a self-corresponding ray *e*. Therefore they are in perspective. Therefore *lm*, *dp*, *qf*, the points of intersection of corresponding rays, are collinear, *i.e.* *l*, *m*, *n* are concurrent.

There are fifty-nine other cases of Pascal's Theorem, and the same number of Brianchon's Theorem. The figure on





the left above illustrates one of these cases of Pascal's Theorem, and the figure on the right one of Brianchon's Theorem. The student should draw figures for a few of

the other cases. He should also write out the proof of Brianchon's Theorem, using the ordinary uniliteral notation for points and biliteral notation for lines. Particular cases of the theorems will be given in the exercises.

Cor. — The triangle determined by three consecutive or three alternate $\left\{\begin{matrix}\text{sides}\\\text{vertices}\end{matrix}\right\}$ of a simple hexagon $\left\{\begin{matrix}\text{inscribed in}\\\text{circumscribed to}\end{matrix}\right\}$ a circle is in perspective with the triangle determined by the remaining three $\left\{\begin{matrix}\text{sides}\\\text{vertices}\end{matrix}\right\}$.

CHAPTER IX.

INVOLUTION.

(114) *Defs.*—If O, A, A′, B, B′, C, C′ . . . . are collinear points, such that $OA.OA' = OB.OB' = OC.OC' = \ldots\ldots = k$, a constant, then the points AA′BB′CC′. . . are said to be *in involution*.

O is called the *centre*, and k the *constant* of the involution.

A and A′, B and B′, C and C′ are called *conjugate points* of the involution.

O and the point at infinity on the range are evidently conjugate points.

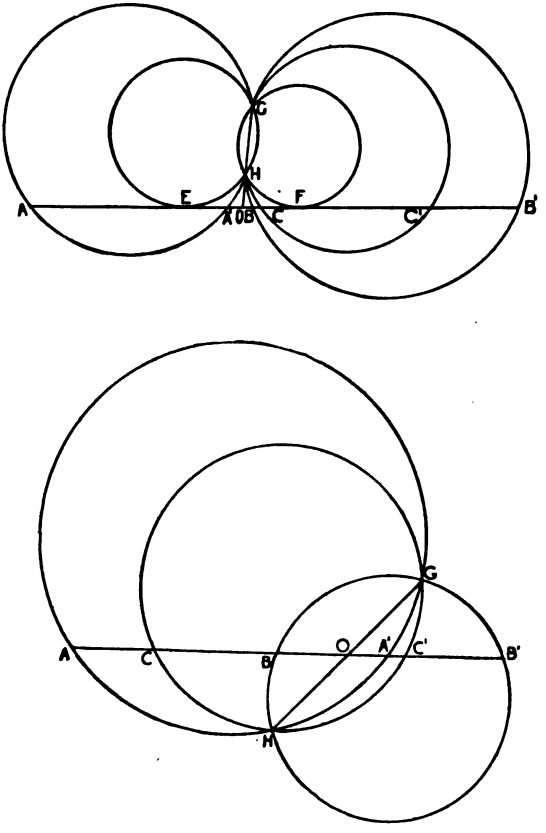
Two points, E and F, situated on the range on opposite sides of O, such that $OE^2 = OF^2 = OA.OA' = OB.OB' = \ldots\ldots = k$, are called the *double points* of the involution. It is evident (and hence the name) that each double point is its own conjugate.

When the constant k is positive, each point lies on the same side of O as its conjugate, and of two pairs of conjugates one pair lies wholly inside or wholly outside the segment joining the other pair.

When k is negative, each point lies on the opposite side of O from its conjugate, and of two pairs of conjugates one pair lies partly within and partly without the segment joining the other pair.

When k is positive, the double points are real; when k is negative, they are imaginary.

(115) PROBLEM.—*Given two pairs of conjugate points in an involution,* A, A′ *and* B, B′, *to find the centre, the conjugate of any other given point* C *on the range, and the double points.*



Construction.—Take any point G outside the axis of the range. Draw two circles round GAA′ and GBB′ intersecting in a second point H. Join GH, and let it cut

the axis in O. O is the centre. Draw a circle round GHC cutting the axis in a second point C'. C' is the conjugate of C. Through G and H draw if possible two circles to touch the axis in E and F. E and F are the double points.

Proof.—By *Euc.* iii. 35, 36.—

$OG.OH = OE^2 = OF^2 = OA.OA' = OB.OB' = OC.OC'$.

If of the pair of conjugates B, B' one lies within and the other without the segment AA', G and H lie on opposite sides of the axis. In this case circles cannot be drawn through G and H to touch the axis; consequently the double points are imaginary.

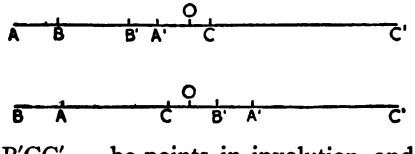
(116) THEOREM.—*The double points of an involution are harmonic conjugates with respect to every pair of conjugate points of the involution.*

Proof.—If AA'BB'CC'. . . be points in involution, E, F the double points, and O the centre of the involution, O is the middle point of EF, and $OE^2 = OF^2 = OA.OA' = OB.OB' = OC.OC' =$ &c. Therefore (*Art.* 60), EFAA', EFBB', EFCC', &c., are harmonic ranges.

Converse.—Pairs of collinear points which are harmonic conjugates with respect to two fixed points on the range are in involution, the fixed points being the double points.

Proof.—Let A and A', B and B', C and C', &c., be the pairs of collinear points, E and F the fixed points, and O the middle point of EF. Then EFAA', EFBB', EFCC', &c., are harmonic ranges. Therefore (*Art.* 60) $OE^2 = OF^2 = OA.OA' = OB.OB' = OC.OC' =$ &c.

(117) THEOREM.—*The points of an involution and their conjugates form two homographic ranges.*



Proof.—Let AA'BB'CC'. . . be points in involution, and O the centre. Then $OA.OA' = OB.OB' = OC.OC' = \ldots = k$.

Hence

$$\frac{AC}{CB} : \frac{AA'}{A'B} = \frac{AO + OC}{CO + OB} : \frac{AO + OA'}{A'O + OB} = \frac{\frac{k}{A'O} + \frac{k}{OC'}}{\frac{k}{C'O} + \frac{k}{OB'}} : \frac{\frac{k}{A'O} + \frac{k}{OA}}{\frac{k}{AO} + \frac{k}{OB'}}$$

$$= \frac{A'O + OC'}{\frac{A'O \cdot OC'}{C'O + OB'}} : \frac{A'O + OA}{\frac{A'O \cdot OA}{AO + OB'}} = \frac{A'O + OC'}{C'O + OB'} : \frac{A'O + OA}{AO + OB'}$$

$$= \frac{A'C'}{C'B} : \frac{A'A}{AB'}.$$

Therefore (ABCA') = (A'B'C'A).

Similarly every other four points (not consisting of two pairs of conjugates) and their conjugates may be proved to form two equianharmonic ranges. It is obvious that four points consisting of two pairs of conjugates and their conjugates form two equianharmonic ranges.

Cor.—The cross ratio of any four points of an involution is equal to the cross ratio of their four conjugates.

Converse.—If in a range of points which are regarded as corresponding in pairs, the points and their respective correspondents form two homographic ranges, the points are in involution.

Proof.—Take as points of reference A and A', B and B', two pairs of corresponding points on the range. Let C and C' be any other pair of corresponding points. Find (*Art.* 115) a point C'' on the range such that AA'BB'CC'' shall be in involution. Then (ABCA') = (A'B'C''A). But by hypothesis (ABCA') = (A'B'C'A). Therefore (A'B'C''A) = (A'B'C'A). Therefore C' coincides with C''. Therefore AA'BB'CC' are in involution, C and C' being conjugate points in the involution.

Similarly it may be proved that any other pair of corresponding points are conjugate points in the involution.

Cor. i.—Six collinear points, which are regarded as corresponding in pairs, are in involution when the cross ratio of four of them (not being two pairs of corresponding

points) equals the cross ratio of the four corresponding points.

Otherwise.—If two equianharmonic ranges have two points in common, either of which, to whichever range it be regarded as belonging, corresponds to the other, the six points involved are in involution.

Cor. ii.—If two homographic ranges have two points in common, either of which, to whichever range it may be regarded as belonging, corresponds to the other, all the points involved are in involution.

Otherwise.—When two homographic ranges on the same axis are such that any one point on the axis has the same point corresponding to it, to whichever range it may be regarded as belonging, every point on the axis possesses this property; and any series of points on the axis and their correspondents are in involution.

Proof.—Let A and B be the two points common to the collinear homographic ranges u and u'. Let C, D be *any* two points in the axis regarded as belonging to the range u, and let E, F be their respective correspondents in the range u'. Then (ABCD) = (BAEF). Therefore (*Cor.* i.) ABCEDF are in involution. Therefore, by the *Cor.* to the theorem of this article (ABEF) = (BACD). That is, E, F, regarded as belonging to the range u, correspond respectively to C, D in the range u'.

(118) THEOREM.—*If a pencil is cut by any one transversal in points in involution, then all transversals cut it in points in involution.*

Proof.—

Let the transversal u cut the pencil in AA'BB'CC' . . . points in involution, and let the transversal u' cut the pencil in the coradial points* $\alpha\alpha'\beta\beta'\gamma\gamma'$. . . Then (AA'BC . . .) = (A'AB'C' . . .) But by *Art.* 99, (AA'BC . . .) = ($\alpha\alpha'\beta\gamma$. . .), and (A'AB'C . . .) = ($\alpha'\alpha\beta'\gamma'$). Therefore ($\alpha\alpha'\beta\gamma$ . . .) = ($\alpha'\alpha\beta'\gamma'$. . .). Therefore, by *Art.* 117, $\alpha\alpha'\beta\beta'\gamma\gamma'$. . . are in involution.

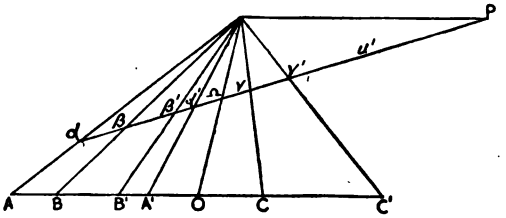
* *i.e.* points lying on the same rays.

Cor. i.—Conjugate points on u' are coradial with conjugate points on u.

Cor. ii.—The double points on u' are coradial with the double points on u.

Cor. iii.—The centre of involution of u' is not in general coradial with the centre of involution of u.

Proof.—If Ω be the point on u' coradial with O, the centre of involution of AA'BB'CC' . . ., and P be the point of intersection of u' with the ray parallel to u, then, by *Cor.* i., since O and the point at infinity on u are conjugates, and P is the point on u' coradial with the point at infinity on u, Ω and P are conjugate points on the involution $\alpha\alpha'\beta\beta'\gamma\gamma'$. . . . Therefore Ω is not the centre of involution of $\alpha\alpha'\beta\beta'\gamma\gamma'$ . . . unless P is the point at infinity on u', *i.e.* unless u' is parallel to u.



Cor. iv.—Ranges projective with a range of points in involution are themselves ranges of points in involution.

Def.—A pencil is said to be *in involution* when any of its sections forms a range of points in involution. Rays which pass through conjugate points of the section are called *conjugate rays;* those which pass through double points of the section are called *double rays* of the pencil.

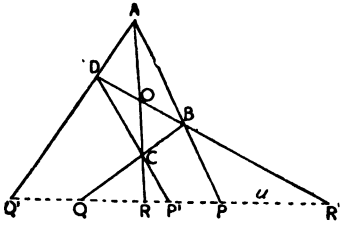
It is evident that the theorems and corollaries of *Arts.* 116 and 117 are true, *mutatis mutandis*, for pencils in involution. We are now in a position to assert (1) that every section or projection of an involution gives another involution, and (2) that involution is only a particular case

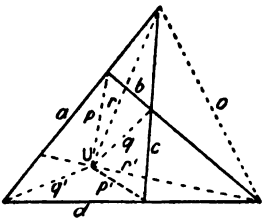
of two homographic ranges having the same axis, or of two homographic pencils having the same vertex. Hence theorems respecting points and pencils in involution are capable of reciprocation in the sense of *Art.* 75.

(119) THEOREMS.—

(α) *The six points in which any transversal cuts the sides of a complete quadrangle form an involution, those points being conjugate which lie on opposite sides.*

(α′) *The six rays which connect any point with the vertices of a complete quadrilateral form an involution, those rays being conjugate which pass through opposite vertices.*





Proof.—

Let ABCD be the quadrangle, and O the point of intersection of the sides AC, BD.

Let the transversal *u* cut the sides AB, CD, BC, DA, AC, BD in the points P, P′, Q, Q′, R, R′ respectively.

Then (QRP′R′) = (BODR′), C being the centre of perspective. But (BODR′) = (PRQ′R′), A being the centre of perspective. Therefore (QRP′R′) = (PRQ′R′) =

Let *abcd* be the quadrilateral, and *o* the connector of the vertices *ac*, *bd*.

Let the point U connect with the vertices *ab*, *cd*, *bc*, *da*, *ac*, *bd* by the lines *p*, *p′*, *q*, *q′*, *r*, *r′* respectively.

Then (*qrp′r′*) = (*bodr′*), *c* being the axis of perspective. But (*bodr′*) = (*prq′r*), *a* being the axis of perspective. Therefore (*qrp′r*) = (*prq′r′*) = (*q′r′pr*). There-

(Q'R'PR). Therefore, by *Art.* 117, *Conv. Cor.* i., PP'QQ'RR' form an involution.

fore, by *Art.* 117, *Conv. Cor.* i., *pp'qq'rr'* form an involution.

PROBLEMS.—

(α) *Given P, P' and Q, Q' two pairs of conjugate points of an involution, and R any other point on the range, to find by means of the ungraduated ruler only, R', the conjugate of R.*

(α') *Given* p, p' *and* q, q' *two pairs of conjugate rays of an involution, and* r *any other ray of the pencil, to construct by means of the ungraduated ruler only,* r', *the conjugate of* r.

Solution.—

Let A and C be the respective points of intersection of any straight line through R with any two straight lines through P and P'. Let B and D be the points of intersection of the connectors QC, AP and Q'A, P'C respectively. Then, by the theorem, R', the point of intersection of *u* the axis of the range, and the connector BD, is such that PP'QQ'RR' are in involution.

Let *a* and *c* be the respective connectors of any point on *r* with any two points on *p* and *p'*. Let *b* and *d* be the connectors of the points of intersection *qc*, *ap* and *q'a*, *p'c* respectively. Then, by the theorem, *r'*, the connector of U the vertex of the pencil, and the point of intersection *bd*, is such that *pp'qq'rr'* are in involution.

The above solution demonstrates even more clearly than *Arts.* 117 and 118 that points and pencils in involution may be regarded as belonging to the geometry of position.

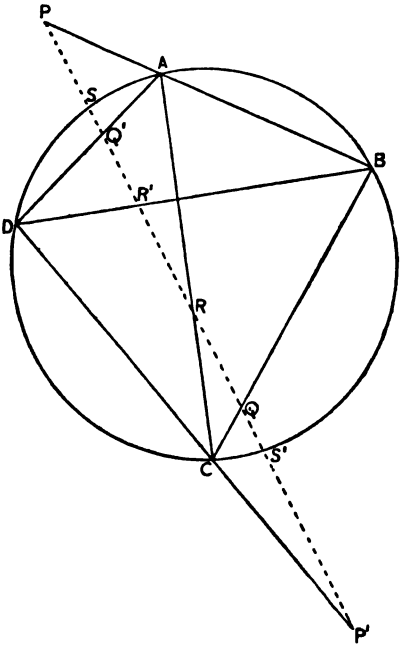
(120) THEOREMS.—

(α) *The two points in which a transversal cuts a circle are conjugate points of the involution formed by the*

(α') *The two tangents drawn from any point to a circle are conjugate rays in the involution formed by the lines*

points in which the transversal cuts the sides of a complete quadrangle inscribed in the circle.

joining the point to the vertices of a complete quadrilateral circumscribed to the circle.



Proof of α.—Let the transversal cut the circle in S, S', and the sides of the inscribed quadrangle ABCD in the same points as in the figure of *Art.* 119.

Then A(PSQ'S') = A(BSDS') = C(BSDS') = C(QSP'S'). Therefore (PSQ'S') = (QSP'S') = (P'S'QS). Therefore PP'QQ'SS' are in involution. (*Art.* 117, *Conv. Cor.* i.).

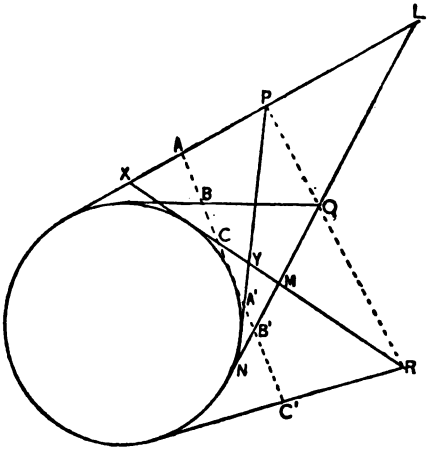
The proof of α' (exactly reciprocal) is left as an exercise to the student.

(121) THEOREMS.—

(α) *If the intersections of pairs of tangents to a circle are collinear, their intersections with any other tangent to the circle form a system of points in involution.*

(α′) *If the connectors of pairs of points on a circle are concurrent, their connectors with any other point on the circle form a pencil in involution.*

Proof of α.—

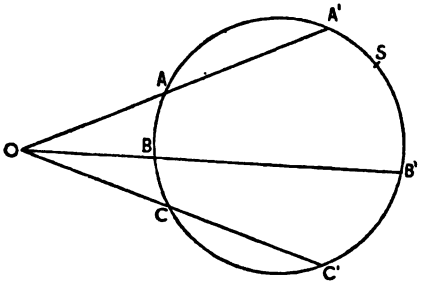


Let three pairs of tangents to a circle intersect collinearly in P, Q, R. Let another tangent cut these pairs in A, A′, B, B′, C, C′. Let the tangents through P, Q, R further intersect each other in L, M, N, X, Y as indicated in the figure. Then, by *Art.* 111, α′, (ABCA′) = (LQMN). (LQMN) = (XRMY), P being the centre of perspective. By *Art.* 111, α′, (XRMY) = (AC′B′A′) = (A′B′C′A). Therefore (ABCA′) = (A′B′C′A). Therefore by *Art.* 117, *Conv. Cor.* i., AA′BB′CC′ are in involution.

Proof of α′.—

Let the connectors of three pairs of points A and A′,

B and B′, C and C′ on a circle be concurrent, and let S be any other point on the circle. Then by *Art.* 111, α, S (ABCA′) = C′(ABCA′). C′(ABCA′) = B (AC′B′A′), AA′



being the axis of perspective. By *Art.* 111, α, B (AC′B′A′) = S (AC′B′A′) = S (A′B′C′A). Therefore S (ABCA′) = S (A′B′C′A). Therefore by *Art.* 117, *Conv. Cor.* i., S. AA′BB′CC′ is a pencil in involution.

The student should as an exercise write out a proof of α' exactly reciprocal to that given for α.

Cors.—

(α) If a simple hexagon is inscribed in one circle, and circumscribed to another, a tangent to the inner circle is cut by the sides in six points in involution.

Proof.—

Since the hexagon is inscribed in a circle, its opposite sides intersect collinearly (Pascal's Theorem), *i.e.* three pairs of tangents to the inner circle intersect collinearly. Therefore any other tangent

(α') If a simple hexagon is circumscribed to one circle, and inscribed in another, a point on the outer circle connects with the vertices in a pencil in involution.

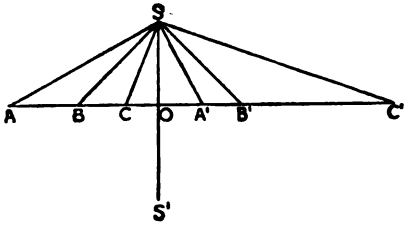
Since the hexagon is circumscribed to a circle, its opposite vertices connect concurrently (Brianchon's Theorem), *i.e.* three pairs of points on the outer circle connect concurrently. Therefore

to the inner circle is cut by them in points in involution.

any other point on the outer circle connects with them in a pencil in involution.

(122) THEOREM.—*A pencil which consists of pairs of rays at right angles is a pencil in involution.*

Proof.—



Let S be the vertex of the pencil, and let SA, SA′; SB, SB′; SC, SC′, &c., be pairs of rays at right angles. Let the pencil be cut by a transversal in A, A′, B, B′, C, C′, &c. Let SO be drawn perpendicular to the transversal and produced to S′, so that S′O = OS. Then circles described on AA′, BB′, CC′, &c., as diameters pass through S and S′. Therefore by *Art.* 115 (see also *Art.* 140) AA′BB′CC′ . . . are in involution. Therefore S.AA′BB′CC′ . . . is a pencil in involution.

Cor.—A pencil in involution which has two pairs of conjugate rays at right angles has every pair of conjugate rays at right angles.

Proof.—Let SA, SA′ and SB, SB′ be two pairs of conjugate rays at right angles, and let SC, SC′ be any other pair of conjugate rays. If SC′ is not perpendicular to SC, let SC″ be perpendicular to SC. Then, S.AA′BB′CC″ and S.AA′BB′CC′ are both pencils in involution. Therefore SC has two conjugates in the involution, determined by SA, SA′, SB, SB′. But this is impossible (*Art.* 114). Therefore SC″ coincides with SC′, which is consequently perpendicular to SC.

CHAPTER X.

ORTHOGONAL CIRCLES. RADICAL AXIS AND CENTRE. COAXAL CIRCLES.

(123) THEOREM. — *If two circles cut each other, and tangents be drawn to each at both points of intersection, the angle between the tangents at one point of intersection is equal to the angle between the tangents at the other point of intersection.*

This can be easily proved by *Euc.* i. 8.

Cor.—When two circles cut each other so that the tangents at one of the points of intersection are at right angles, the tangents at the other point of intersection are also at right angles.

Def.—Two circles which cut each other so that the tangents at the points of intersection are at right angles, are said to be *orthogonal* or to cut each other *orthogonally.*

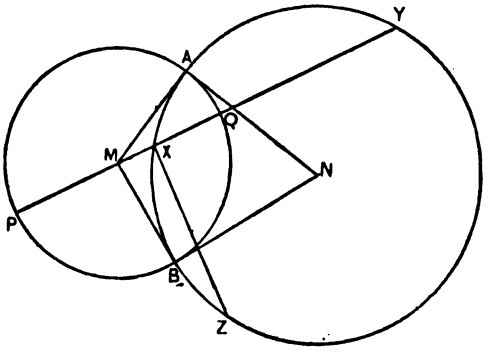
(124) THEOREM. — *If two circles are orthogonal, the tangent to either at a point of intersection passes through the centre of the other, and conversely the radius of either drawn to a point of intersection is a tangent to the other.*

The first part of the statement follows from *Euc.* iii. 19, and the second from *Euc.* iii. 18.

Converse.—If two circles intersect in such a manner that the straight line joining a point of intersection with the centre of one of them is a tangent to the other, the circles are orthogonal.

Cor.—If two circles are orthogonal, the straight line joining their centres subtends a right angle at either point of intersection; and conversely, if two circles intersect in such a manner that the straight line joining their centres subtends a right angle at either point of intersection, the circles are orthogonal.

PROBLEM.—*With a given point as centre, to draw a circle orthogonal to a given circle.*



Solution.—Let N be the given point, and M the centre of the given circle. From N draw a tangent to the circle M* touching it at A. With N as centre and NA as radius describe a circle. Since NA is a radius of circle N and a tangent to circle M, the circles are orthogonal (*Converse* above).

If N lies on the circle M, NA is infinitely small, and therefore the circle N coincides with the point N. If N lies inside the circle M, NA is imaginary, and therefore the circle N is imaginary.

(125) THEOREM.—*A transversal drawn across two orthogonal circles, through the centre of either, is cut harmonically by the circles.*

* "Circle M" is a contraction for the "circle whose centre is M."

Proof.—Let the circles M and N (see fig. of *Art.* 124) intersect orthogonally at A. Let a transversal through the point M cut the circle M in P, Q and the circle N in X, Y. Join MA.

Then since MA is a tangent to circle N, $MX.MY = MA^2 = MP^2 = MQ^2$. $\therefore$ PQXY is a harmonic range.

Converse.—If four points form a harmonic range, the circle described on the segment between either pair of harmonic conjugates as diameter cuts orthogonally every circle passing through the other pair of harmonic conjugates.

Cor.—A circle is cut by the polar of any point on it, with respect to an orthogonal circle, in points which lie on the straight lines joining the point with the centres of the two circles.

Proof.—At X draw a perpendicular to XY, meeting circle N at Z. Then since $MX.MY = MQ^2$, XZ is the polar of Y on the circle N with respect to the circle M. But X lies on YM, and since YXZ is a right angle, Z lies on YN.

PROBLEM.—*Through a given point to draw a circle orthogonal to a given circle.*

Solution.—Let X be the given point and M the centre of the given circle. Through X and M draw a straight line cutting the circle M at P and Q. Find Y the harmonic conjugate of X with respect to P and Q. Any circle through X and Y cuts the given circle orthogonally.

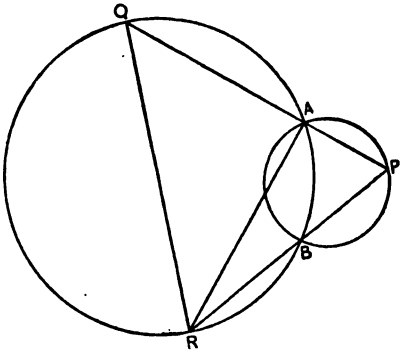
(126) THEOREM.—*If two circles are orthogonal, the angles in the segments on opposite sides of the common chord are complementary.*

Proof.—Let the circles M and N (see fig. of *Art.* 124) intersect orthogonally at A and B. Then in the quadrilateral MANB, the angles A and B are each equal to a right angle. $\therefore$ the angles M and N are together equal to two right angles. $\therefore$ the angles at the circumferences are together equal to one right angle.

Converse.—Two intersecting circles are orthogonal if the angles in the segments on opposite sides of the common chord are complementary.

Cor.—If two circles are orthogonal, any straight line through a point of intersection and terminated by the circumferences, subtends a right angle at the other point of intersection; and conversely.

(127) THEOREM.—*Two straight lines drawn from any point on the circumference of one of two orthogonal circles, through the points of intersection, meet the other circle in points which are the extremities of a diameter.*



Proof.—From P on one of the circles, through the points of intersection A and B, draw straight lines meeting the other circle at Q and R respectively. Join RA, QR. Then, by *Art.* 126, *Cor.*, the angles at A are right. ∴ QR is a diameter of the circle QAR.

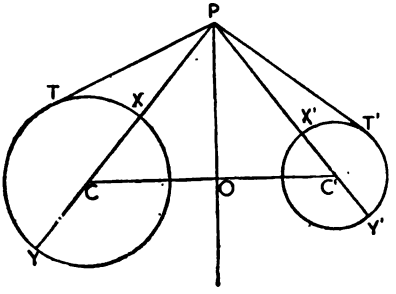
Cor.—The connectors QB, RA intersect on the circle PAB.

Converse.—The locus of the points of intersection of the straight lines joining two fixed points on a circle to the extremities of a variable diameter is the circle through the fixed points orthogonal to the original circle.

This furnishes us with the following solution of the problem of *Art.* 125. Let P be the given point, and QR any diameter of the given circle. Join PQ, PR, cutting the circle in A and B. The circumcircle of ABP is orthogonal to the given circle.

(128) *Def.*—The locus of a point P such that when secants PXY, PX'Y' are drawn from it to two circles, PX.PY = PX'.PY', is called the *radical axis* of the circles.

THEOREM.—*The radical axis of two circles is a straight line perpendicular to the straight line joining the centres of the circles, and dividing this line, so that the difference of the squares on the two parts is equal to the difference of the squares on the radii of the circles.*



Proof.—Let P be a point on the radical axis, and C, C' the centres of the circles. Join PC, PC', CC'. Let PC meet circle C in X, Y, and PC' meet circle C' in X', Y'. Draw PO perpendicular to CC'.

Then by *Euc.* iii. 35 or 36, $PX.PY = CP^2 - CX^2$, and $PX'.PY' = C'P^2 - C'X'^2$. $\therefore CP^2 - CX^2 = C'P^2 - C'X'^2$. $\therefore CX^2 - C'X'^2 = CP^2 - C'P^2 = (Euc.\ i.\ 47)\ CO^2 - C'O^2 = CC'(CO - OC')$. $\therefore CO - OC' =$ a constant. $\therefore$ O is a fixed point. $\therefore$, since COP is a right angle, PO is the locus of P.

(129) THEOREMS AND PARTICULAR CASES.—

(i) *Tangents drawn from any point on the radical axis of two circles to the circles are equal.*

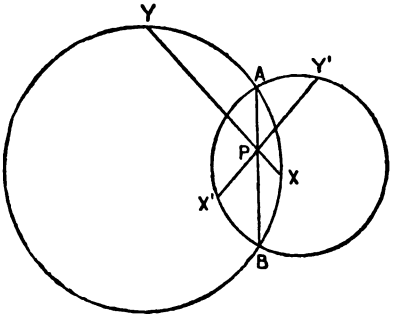
Proof.—Let P (see fig. of *Art.* 128) be the point, and PT, PT′ the tangents to the circles C and C′ respectively. Then $PT^2 = PX.PY = PX'.PY' = PT'^2$. $\therefore PT = PT'$.

Converse.—The locus of a point from which tangents drawn to two circles are equal is the radical axis of the circles.

(ii) *The common tangents (where they exist) of two circles are bisected by the radical axis.*

(iii) *The locus of the centre of a variable circle which cuts two given circles orthogonally is the radical axis of the given circles.*

(iv) *If two circles intersect, their radical axis is the common chord.*



Proof.—Through P any point on the common chord AB draw the secants XY, X′Y′ to the two circles. Then $PX.PY = PA.PB = PX'.PY'$. $\therefore$ P is a point on the radical axis.

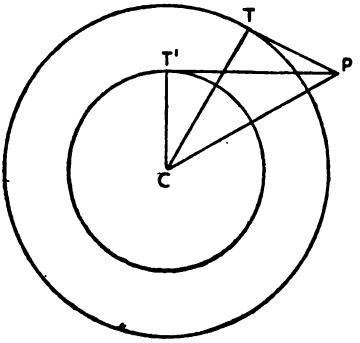
In this case the tangents drawn to the circles from points on the finite segment AB are imaginary.

(v) *If two circles touch each other, their radical axis is the tangent at the point of contact.*

(vi) *If two circles neither intersect nor touch each other, their radical axis is external to both.*

In the cases referred to in iv. and v. the circles meet in two real points (which may be distinct or coincident) on their radical axis; in the present case the circles may be regarded as meeting in two imaginary points on their radical axis.

(vii) *If two circles are concentric, their radical axis is the straight line at infinity.*



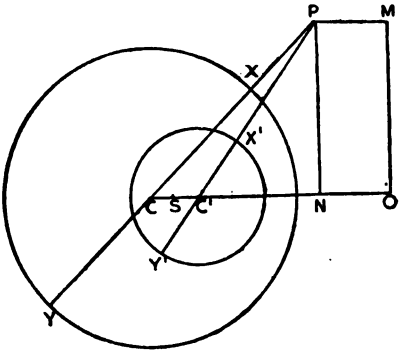
Proof.—Let C be the common centre of two concentric circles, and T any point on the greater. Draw the tangent at T, and let P be a variable point on it. Draw PT′ a tangent to the smaller circle, and join CP, CT, CT′.

Then, since angles CTP, CT′P are each equal to a right angle, T and T′ lie on the circle having CP as diameter. ∴ PT′ is less than PC and greater than PT (*Euc.* iii. 7). Again, angle PCT = angle PTC (a constant) − angle CPT. But as P moves off from T, angle CPT diminishes, till in the limit when P is at an infinite distance

from T, it vanishes altogether. ∴ when P is the point at infinity on the tangent at T, PT = PC, and ∴, since PT′ is intermediate in length between PT and PC, PT = PT′. ∴ the point at infinity on the tangent at T lies on the radical axis of the circles. Similarly the point at infinity on any other tangent to the larger circle lies on the radical axis. ∴ the radical axis is the straight line at infinity.

(viii) *If two circles are equal, their radical axis is equidistant from their centres; if they are unequal and not concentric, their radical axis is nearer to the centre of the smaller circle than to the centre of the greater.*

(130) THEOREM.—*If through any point* P *secants* PXY, PX′Y′ *are drawn to two circles whose centres are* C *and* C′ *respectively, and if* PM *is the perpendicular from* P *on the radical axis of the circles,* PX.PY − PX′.PY′ = 2MP.CC′.



Proof.—Join PC, PC′, CC′. Let S be the middle point of CC′, O the point of intersection of CC′ with the radical axis. Draw PN perpendicular to CC′.

Then, by *Euc.* iii. 35, 36, $PX.PY - PX'.PY'$.

$= PC^2 - CX^2 - (PC'^2 - C'X'^2)$

$= PC^2 - PC'^2 - (CX^2 - C'X'^2)$.

$= CN^2 - C'N^2 - (CO^2 - C'O^2)$

$= (CN + C'N)(CN - C'N) - (CO + C'O)(CO - C'O)$.

$= 2SN.CC' - 2SO.CC' = 2ON.CC' = 2MP.CC'$.

Cors. and Particular Cases.—

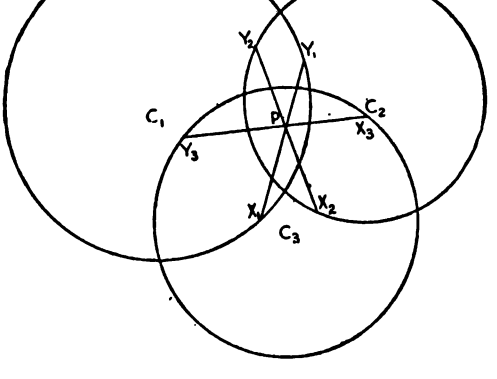
(i) When P is on the radical axis we obtain the property by which the radical axis is defined.

(ii) If P is on the circle C', $PX'.PY'$ vanishes, and we have $PX.PY = 2MP.CC'$. If P is on the circle C, $PX'.PY' = 2PM.CC'$.

(iii) If PT, PT' are tangents from P to the circles C, C', $PT^2 - PT'^2 = 2MP.CC'$.

(iv) If P is on the circle C', $PT^2 = 2MP.CC'$; if on the circle C, $PT'^2 = 2PM.CC'$.

(131) THEOREM.—*The three radical axes of three circles taken in pairs are concurrent.*



Proof.—Let C_1, C_2, C_3 be the three circles, and let P be the point of intersection of the radical axes of the pairs

C_1, C_2 and C_2, C_3. Let PX_1Y_1, PX_2Y_2, PX_3Y_3 be secants through P to C_1, C_2, C_3 respectively.

Then $PX_1.PY_1 = PX_2.PY_2$, and $PX_2.PY_2 = PX_3.PY_3$. $\therefore$ $PX_1.PY_1 = PX_3.PY_3$. $\therefore$ P is a point on the radical axis of C_1, C_3. $\therefore$ the three radical axes meet in P.

Def.—The point of concurrence of the three radical axes of three circles taken in pairs is called the ***radical centre*** of the circles.

Cors. and Particular Cases.—

(i) A point P is the radical centre of three circles, if when secants PX_1Y_1, PX_2Y_2, PX_3Y_3 are drawn to the circles, $PX_1.PY_1 = PX_2.PY_2 = PX_3.PY_3$.

(ii) When the centres of three circles are collinear, their radical centre is at infinity.

(iii) When three circles have a common radical axis (as when the circles are concentric or touch at a point), any point on the radical axis is the radical centre.

This case is more fully presented in *Art.* 135, xi.

(iv) When three circles intersect at a common point, this point is their radical centre.

(v) When three circles intersect in such a manner that there is an area common to all three, the radical centre is situated within this area.

(vi) When three circles have neither a common point nor a common area, the radical centre lies outside the circles.

(vii) When the radical centre of three circles is external to them, equal tangents can be drawn from it to the circles. With one of these tangents as radius, and the radical centre as centre, a circle orthogonal to the three circles can be drawn. When the radical centre is at infinity, the orthogonal circle coincides with the straight line joining the centres of the three circles. When the radical centre lies on each of the three circles, the

tangents from it to the circles are infinitely small, and consequently the orthogonal circle coincides with the radical centre. When the radical centre lies inside each of the three circles, the tangents from it to the circles are imaginary, and consequently the orthogonal circle is imaginary. In all cases the radius of the orthogonal circle is equal to $\sqrt{PX.PY}$, where P is the radical centre and PXY a secant to one of the circles.

(viii) When three circles touch or intersect in pairs, the tangents at the points of contact and the chords of intersection are concurrent.

(132) PROBLEM.—*To draw the radical axis of two given circles.*

Solution.—If the circles meet at a point, the common tangent at the point or the common chord is the radical axis.

If the circles do not meet, let them be denoted by C_1, C_2. Draw a third circle C_3 to cut C_1 and C_2. Let the common chord of C_1, C_3 and the common chord of C_2, C_3 intersect at P. The perpendicular from P to the line of centres of C_1, C_2 is obviously the radical axis of C_1, C_2.

(133) THEOREM.—*If any three straight lines be drawn from the vertices of a triangle to the opposite sides, and circles be described on them as diameters, the orthocentre of the triangle is the radical centre of the circles.*

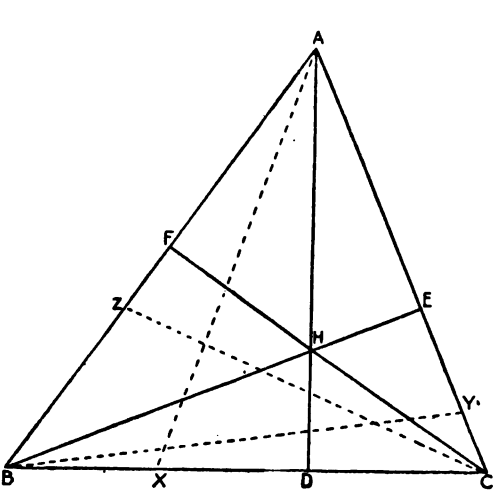
Proof.—Let ABC be a triangle, AD, BE, CF its altitudes, H its orthocentre, and AX, BY, CZ any three straight lines drawn from the vertices to meet the opposite sides.

Then, since the angles at D, E, F are right, AD, BE, CF are respectively chords of the circles whose diameters are AX, BY, CZ. But HA.HD = HB.HE = HC.HF. ∴ H is the radical centre of the three circles (*Art.* 131, i).

Cor. i.—The circles described on the sides of a triangle as diameters have the orthocentre of the triangle as their radical centre.

Cor. ii.—The circle with respect to which ABC is self-conjugate, cuts orthogonally the circles whose diameters are AX, BY, CZ.

Proof.—H is the centre of the circle with respect to



which ABC is self-conjugate, and its radius is equal to $\sqrt{HA.HD}$ (*Art.* 85). But H is the radical centre of the circles whose diameters are AX, BY, CZ, and HAD is a secant through one of them. ∴ the former circle is orthogonal to the latter circles (*Art.* 131, vii).

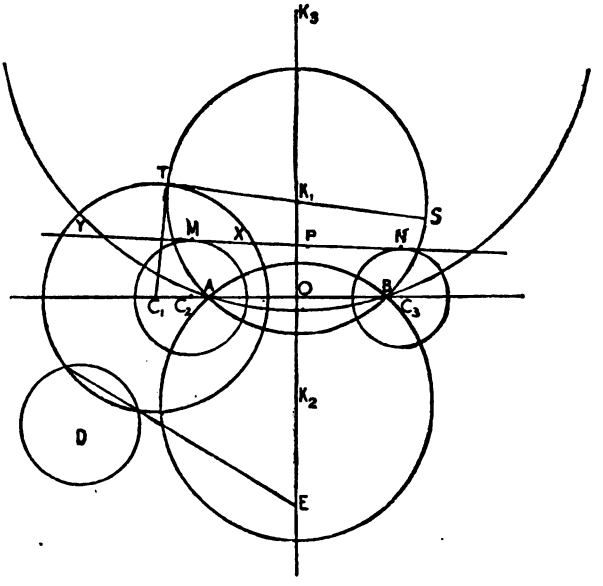
The sides of the triangle might be substituted for AX, BY, CZ.

(134) PROBLEM.—*To construct systems of circles such that all the pairs in a system shall have the same radical axis.*

(*a*) Let A and B be any two points. Through A and B

draw any number of circles. These circles form a system for which the common chord AB (*Art.* 129, iv) is the radical axis of every pair.

(*b*) Draw a straight line bisecting AB perpendicularly in O. This is obviously the line of centres of the system (*a*). Let K_1 be a point on it. Construct (*Art.* 124) a system of circles having their centres collinear with A and B, and



cutting orthogonally any given circle passing through A and B, say the circle whose centre is K_1. Then (*Art.* 129, iii) K_1 is a point on the radical axis of every pair of circles of the system. And K_1O is perpendicular to their line of centres. $\therefore$ K_1O is the radical axis of every pair of circles of the system.

Those circles of the system (*b*) whose centres lie between

A and B are imaginary; and the circles whose centres are A and B are infinitely small and coincide with A and B (*Art.* 124).

Defs.—If a system of circles be such that every pair has the same radical axis, the circles are said to be *coaxal.* With respect to a system of coaxal circles constructed as in (b), the points (A, B) where the circle (K_1) to which the circles of the system are drawn orthogonal, cuts the line of centres of the circles of the system, are called the *limiting points* of the system. As already seen, the limiting points are infinitely small circles of the system. Coaxal circles constructed as in (b) are called *limiting-point* circles. Coaxal circles constructed as in (a) are called *common-point* circles. When a system of common-point coaxal circles, and a system of limiting-point coaxal circles are so related that the common points of the one system are the limiting points of the other, they are called *conjugate* coaxal systems.

The centres of the two particular systems of coaxal circles drawn are collinear. Since the common radical axis of every coaxal system is perpendicular to the straight line joining every pair of centres of the system, it follows that the centres of *every* system of coaxal circles are collinear.

Of the two systems drawn, one is a common-point system, the other a limiting-point system. It will now be shown that *every* system of coaxal circles is either a common-point or a limiting-point system. This proposition is included in the following theorem: In every system of coaxal circles, the circles (α) intersect at two common points and have no limiting points, or (β) meet at two coincident points which may be regarded either as common or limiting, or (γ) do not meet at all, and thus have no common points, but have two limiting points.

Proof.—Let C_1, C_2 be two circles, and C_3 any third circle of the system.

Then C_1, C_2 must (α) intersect at two points, or (β) meet at two coincident points, or (γ) not meet at all.

(α) If C_1, C_2 intersect at A and B, AB is the radical axis of the system, and consequently of C_1, C_3. $\therefore$ C_1, C_3 intersect at A and B. $\therefore$ all the circles intersect at A and B. $\therefore$ the system has common points. Again the smallest circle of the system has AB as diameter. $\therefore$ there are no limiting points, *i.e.* the limiting points are imaginary.

(β) If C_1, C_2 meet at two points coinciding at A, the tangent to C_1 at A is the radical axis of the system, and consequently of C_1, C_3. $\therefore$ C_1, C_3 touch at A. $\therefore$ all the circles touch at A. The two points coinciding at A are common to all the circles, and are therefore common points; they are also the smallest circles of the system, and are therefore limiting points.

(γ) If C_1, C_2 do not meet at all, their radical axis, which is the radical axis of the system, meets neither. $\therefore$ C_3 does not meet C_1, for if it did, their common tangent or chord would be the radical axis, which would consequently meet C_1. Thus no two circles of the system meet each other. $\therefore$ there are no common points, *i.e.* the common points are imaginary.

Again, if the centres of the circles C_1, C_2 be denoted by C_1, C_2 respectively (see figure), and if K_1 be any point on the radical axis of the system, K_1T a tangent from it to the circle C_1, and O the point of intersection of the radical axis with the common line of centres, $C_1T^2 < C_1O^2$. $\therefore$ $C_1K_1^2 - C_1T^2 > C_1K_1^2 - C_1O^2$. $\therefore$ $K_1T^2 > K_1O^2$. $\therefore$ $K_1T > K_1O$. $\therefore$ a circle described with K_1 as centre and K_1T as radius will cut the line of centres in two points, say A and B. Since $K_1A = K_1B = K_1T$ = a tangent from K_1 to any circle of the system, A and B are infinitely small circles of the system, *i.e.* they are limiting points of the system.

Another particular case which may be mentioned is that of concentric circles. Concentric circles form a limiting-point coaxal system, the radical axis being the straight line at infinity, and the limiting points coinciding with the common centre (*Art.* 129, vii).

(135) Reference should be made in this and in the next five articles to the figure of *Art.* 134.

THEOREMS—

(i) *The radical axis of a system of coaxal circles is the line of centres of the conjugate system.*

(ii) *The radical axis of a system of coaxal circles is one of the circles of the system, viz. the circle whose centre is at an infinite distance from the radical axis.*

(iii) *The distances from the limiting points of any point on the radical axis of a system of limiting-point coaxal circles are equal to each other and to the tangents drawn from the point to the circles of the system.*

Proof.—The limiting points are themselves circles of the system, and the distances referred to are tangents to these circles.

(iv) *A common tangent to any two circles of a limiting-point coaxal system subtends a right angle at either limiting point.*

This follows from the fact (*Th.* iii.) that the point where the common tangent cuts the radical axis is equidistant from the points of contact and from the limiting points.

(v) *Every circle of a coaxal system cuts orthogonally every circle of the conjugate system.*

Proof.—The tangents drawn from the centre K_2 of any circle of the common-point system to the circles of the limiting-point system are equal to the distances of K_2 from the limiting points, *i.e.* they are radii of the circle K_2. $\therefore$ the circle K_2 is orthogonal to all the circles of the limiting-point system. The same holds for every other circle of the common-point system.

(vi) *If a circle is orthogonal to two circles of a coaxal system, it is orthogonal to all the circles of the system* (*Art.* 129, iii).

(vii) *If a circle has its centre on the radical axis of a coaxal system, and cuts orthogonally one of the circles of the system, it is orthogonal to all the circles of the system.*

(viii) *If a system of circles is cut orthogonally by two circles, the system is coaxal.*

Proof.—Let K_1, K_2 be the centres of the two cutting circles. Then the tangents from K_1 to the circles of the system, being radii of the circle K_1, are all equal. Also the tangents from K_2 to the circles are all equal. $\therefore$ K_1K_2 is the radical axis of every pair of circles of the system.

(ix) *If circles have their centres collinear, and are cut orthogonally by a circle, they are coaxal.*

(x) *Circles orthogonal to a coaxal system form the conjugate coaxal system.*

(xi) *Any point on the radical axis of three coaxal circles may be regarded as the radical centre of the circles.* (Compare *Art.* 131, iii).

(xii) *If three circles have more than one radical centre, they are coaxal.*

Proof.—The three radical axes pass through each of the radical centres, and consequently coincide, *i.e.* the circles have a common radical axis.

(xiii) *When circles touch each other at a point, the circles of the conjugate coaxal system touch each other at the same point.*

(xiv) *When circles are concentric, the straight lines through the common centre form the conjugate coaxal system.*

(136) PROBLEMS.—

(i) *To find the limiting points of a limiting-point coaxal system of which two circles are given.*

Solution.—Draw the line of centres and the radical axis. With any point on the latter as centre draw (*Art.* 124) a circle orthogonal to either (and there-

fore by *Art.* 129, iii. to both) of the given circles and cutting the line of centres at A and B. A and B are obviously the limiting points.

(ii) *One circle and a limiting point of a coaxal system being given, to find the other limiting point.*

Solution.—Draw the diameter of the given circle which passes through A, the given limiting point. Through A draw (*Art.* 125) a circle orthogonal to the given circle and cutting the diameter drawn at B. B is obviously the second limiting point.

(137) PROBLEM.—*Through a given point to draw a circle orthogonal to two given circles.*

Solution.—Let T be the given point.

(i) If the circles (say C_2, C_3) do not meet and are not concentric, find the limiting points A, B. The circle described about TAB is the circle required.

(ii) If the circles are concentric, the straight line joining T and the common centre is the circle required.

(iii) If the circles touch, the circle drawn through T to touch their line of centres at the point of contact is the circle required.

(iv) If the circles (say K_2, K_3) intersect at A and B, describe a circle about TAB, and at T draw a tangent to this circle cutting AB at C_1. The circle having C_1 as centre and C_1T as radius is the circle required.

(138) THEOREM.—*The radical axes of a circle and each of the circles of a coaxal system taken in succession are concurrent.*

Proof.—Let D be the given circle, C_1 a certain circle of the coaxal system, and C_2 any other circle of the system. Let the radical axis of D and C_1 cut the common radical axis of the system at E. Then E is the radical centre of D, C_1, C_2. $\therefore$ the radical axis of D and C_2 passes through E.

Similarly, the radical axis of D and any other circle of the system passes through E.

(139) THEOREM.—*The polars of a point with respect to each of the circles of a coaxal system taken in succession are concurrent.*

Proof.—Through T, the given point, draw a circle cutting any two, and therefore all the circles of the system orthogonally. Draw TS a diameter of this circle. Then by *Art.* 125, *Cor.*, the polar of T with respect to every circle of the system passes through S.

(140) THEOREM.—*Coaxal circles cut a transversal in points in involution.*

Proof.—Let the transversal cut the radical axis of the system in P, and any circle of the system in X and Y. Then, by *Art.* 128, PX.PY = a constant. ∴ by *Art.* 114, *Def.*, the points in which the straight line cuts the circles form an involution.

Cor. i.—P is the centre of the involution.

Cor. ii.—If the circles form a limiting-point system, X and Y are on the same side of P, and consequently the involution has double points.

Cor. iii.—If the circles form a common-point system, and the transversal cuts the common chord externally, X and Y are on the same side of P, and consequently the involution has double points.

Cor. iv.—If the circles form a common-point system, and the transversal cuts the common chord internally, X and Y are on opposite sides of P, and consequently the involution has no real double points.

Cor. v.—If the transversal touches two circles of the system at M and N, M and N are the double points.

Proof.—$PM^2 = PN^2 = PX.PY$ (*Art.* 129, i).

Cor. vi.—MNXY is a harmonic range.

Cor. vii.—If the circles form a limiting-point system,

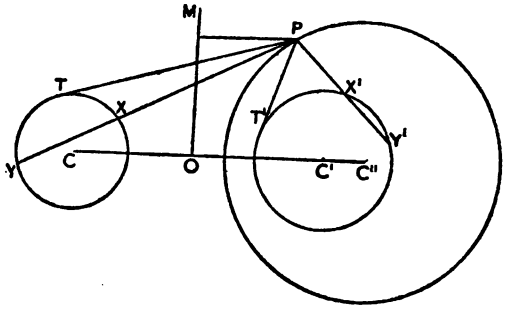
and if A be one of the limiting points, the angle XAY is bisected internally and externally by AM and AN.

Proof.—A.MNXY is a harmonic pencil, and (*Art.* 135, iv) MAN is a right angle. ∴ AM, AN are the bisectors of the angle XAY (*Art.* 66, ii).

Cor. viii.—If the circles form a limiting-point system, the centre of the involution is equidistant from the double points and from the limiting points (*Art.* 135, iii).

Cor. ix.—The line of centres of the limiting-point coaxal circles cuts the circles in an involution whose double points coincide with the limiting points.

(141) THEOREM.—*If* C, C′, C″ *are the centres of three coaxal circles, and if through any point* P *on the circle* C″ *secants* PXY, PX′Y′ *are drawn to the circles* C *and* C′, PX.PY : PX′.PY′ = CC″ : C′C″.



Proof.—Let PM be the perpendicular from P on the common radical axis. Then, by *Art.* 130, ii, since P is on the circle C″, PX.PY = 2MP.CC″, and PX′.PY′ = 2MP.C′C″. Therefore PX.PY : PX′.PY′ = CC″ : C′C″.

Cor. i.—If PT, PT′ are tangents from P to the circles C and C′, $PT^2 : PT'^2 = CC'' : C'C''$.

Cor. ii.—If the circle C″ belongs to a system of which A and B are the limiting points $PA^2 : PB^2 = AC'' : BC''$.

Converses.—If from a variable point P secants PXY and PX'Y' are drawn to two circles, and if P move so that PX.PY : PX'.PY' = a constant, the locus of P is a circle coaxal with the others.

If a point move so that the ratio of the tangents drawn from it to two circles is constant, its locus is a circle coaxal with the others.

If a point move so that the ratio of its distances from two fixed points is constant, its locus is a circle belonging to the coaxal system of which the fixed points are the limiting points.

(142) THEOREM.—*If* C, C', C'' *are the centres of three coaxal circles whose radii are* r, r', r'' *respectively, then*—

$$r^2.C'C'' + r'^2.C''C + r''^2.CC' = -C'C''.C''C.CC'.$$

Proof.—Let O be the point of intersection of the line of centres and the common radical axis.

Then $r^2 - r'^2 = CO^2 - C'O^2$, $r'^2 - r''^2 = C'O^2 - C''O^2$.

$\therefore\ r^2 - CO^2 = r'^2 - C'O^2 = r''^2 - C''O^2 = k.$

$\therefore\ r^2.C'C'' + r'^2.C''C + r''^2.CC' = CO^2.C'C'' + C'O^2.C''C + C''O^2.CC' + (C'C'' + C''C + CC')k.$

But by *Art.* 11, $CO^2.C'C'' + C'O^2.C''C + C''O^2.CC' = -C'C''.C''C.CC'$, and by *Art.* 7, $C'C'' + C''C + CC' = 0$.

$\therefore\ r^2.C'C'' + r'^2.C''C + r''^2.CC' = -C'C''.C''C.CC'.$

(143) THEOREM.—*The circles described on the diagonals of a complete quadrilateral as diameters are coaxal.*

Proof.—Let LMNOPQ be a complete quadrilateral constructed as in *Art.* 80. Then LN, OM, PQ may be regarded as lines drawn from the vertices to the opposite sides of the following triangles, viz., LMQ, LOP, MPN, NQO. Therefore, by *Art.* 133 the orthocentres of these triangles are the radical centres of the circles on LN, OM, PQ as diameters. Therefore by *Art.* 135, xii, these circles are coaxal.

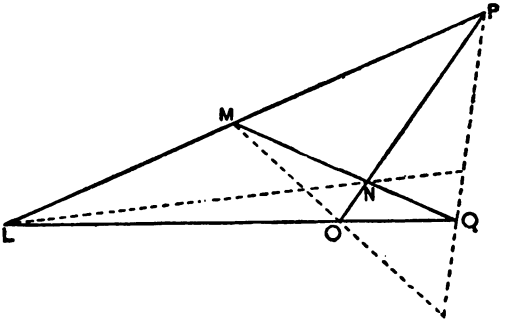
Cor. i.—The middle points of the diagonals of a complete quadrilateral are collinear.

Proof.—The centres of coaxal circles are collinear (*Art.* 134).

Cor. ii.—The orthocentres of the triangles formed by any four straight lines are collinear.

Proof.—By the theorem the orthocentres are radical centres of three coaxal circles. Therefore they lie on the common radical axis.

Cor. iii.—Circles having as diameters the lines con-



necting the vertices of a triangle with the points where the opposite sides are cut by a transversal, are coaxal.

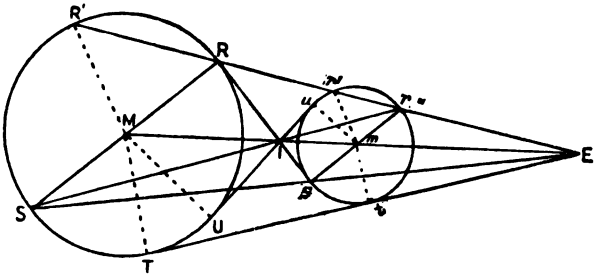
Cor. iv.—The coaxal system conjugate to the circles whose diameters are LN, OM, PQ, is formed by the circles to which the triangles LMQ, LOP, MPN, NQO are self-conjugate.

Proof.—By *Art.* 133, ii, the latter circles cut the former orthogonally. Therefore, by *Art.* 135, x, the two systems are conjugate coaxal systems.

CHAPTER XI.

CENTRES, CIRCLES, AND AXES OF SIMILITUDE OF CIRCLES.

(144) *Def.*—If M, *m* be the centres of two circles, and if M*m* be divided in the points E and I, so that EM : E*m* = − IM : I*m* = radius of circle M : radius of circle *m*, the last ratio being regarded as positive, E is called the *external centre of similitude*, and I the *internal centre of similitude* of the two circles.



THEOREMS.—

(i) MmIE *is a harmonic range* (*Art.* 54, *def.*).

(ii) E *lies outside*, I *inside the segment* Mm (*Art.* 3).

(iii) *When both circles lie wholly outside each other*, E *and* I *lie outside the circles.*

(iv) *When one circle lies wholly within the other*, E *and* I *lie within the smaller circle.*

(v) *When the two circles are concentric,* E *and* I *coincide with the common centre.*

(vi) *When the two circles intersect,* I *lies within,* E *outside both circles.*

(vii) *When the two circles touch externally,* I *coincides with the point of contact, and* E *lies outside both circles.*

(viii) *When the two circles touch internally,* E *coincides with the point of contact, and* I *lies inside both circles.*

(ix) *When the two circles are equal,* E *is at infinity, and* I *is the middle point of* Mm (*Art.* 3).

(145) THEOREM.—*The straight line joining the extremities of any pair of parallel radii of two circles cuts the line of centres in the external centre of similitude if the radii are on the same side of the line of centres, and in the internal centre of similitude if the radii are on opposite sides of the line of centres.*

Proof.—Let MR, *mr* (fig. of *Art.* 144) be two parallel radii of the circles M and *m* respectively, and on the same side of M*m*, and let R*r* cut M*m* in E. Then the triangles MRE, *mr*E are similar. Therefore EM : E*m* = MR : *mr*, and E obviously lies outside the segment M*m*. Therefore E is the external centre of similitude of the circles.

Again, let MS, *mr* be two parallel radii on opposite sides of M*m*, and let S*r* cut M*m* in I. Then the triangles MSI, *mr*I are similar. Therefore IM : I*m* = MS : *mr*, and I obviously lies within the segment M*m*. Therefore I is the internal centre of similitude of the circles.

Cor. i.—The straight lines which join the extremities of parallel diameters of two circles *directly*, meet in the external centre of similitude; the straight lines which join the extremities of parallel diameters of two circles *transversely*, meet in the internal centre of similitude.

Cor. ii.—The direct common tangents (where they exist)

of two circles pass through the external centre of similitude; the transverse common tangents (where they exist) pass through the internal centre of similitude.

Proof.—Let a direct common tangent touch circle M (fig. of *Art.* 144) in T and circle m in t. Join MT, mt. Then the angles MTt, mtT are right. Therefore MT, mt are parallel radii on the same side of Mm. Therefore Tt passes through E.

A similar proof holds for a transverse common tangent Uu.

(146) THEOREM.—(Converse of last theorem).

The four points in which two circles are cut by any transversal drawn through either of their centres of similitude are the extremities of two pairs of parallel radii.

Proof.—Through E, the external centre of similitude of the two circles M and m (fig. of *Art.* 144), draw a transversal cutting the circle m in r, r' and the circle M in R, R'; R' and r' being on concave parts of the circles with reference to E and on the same side of E. Join MR, MR', mr, mr'. Then MR = MR'. Therefore angles MRR', MR'R are each acute. Similarly angles mrr', $mr'r$ are each acute. It follows that angles MRE, mrE are of the same species. In the triangles MRE, mrE, the angle at E is common, EM : MR = Em : mr, and angles MRE, mrE are of the same species. Therefore, by *Euc.* vi. 7, angles RME and rmE are equal. Therefore MR is parallel to mr.

Again, in the triangles MR'E, mr'E, the angle at E is common, EM : MR' = Em : mr', and angles MR'E, mr'E are acute. Therefore, by *Euc.* vi. 7, angles R'ME and $r'm$E are equal. Therefore MR' is parallel to mr'.

The theorem may be similarly proved for the internal centre of similitude.

Def.—Two points in which two circles are cut by a transversal drawn through a centre of similitude are said to be *corresponding* (or *homologous*) points with reference to

that centre, if they are the extremities of two parallel radii; otherwise they are said to be *non-corresponding* (or *anti-homologous*).

By making the various possible figures it will be observed that corresponding points with reference to E (external centre of similitude) lie on the same side of E, and corresponding points with reference to I (internal centre of similitude) lie on opposite sides of I, while non-corresponding points with reference to E lie on the same side or opposite sides of E according as E lies outside or inside the two circles, and non-corresponding points with reference to I lie on the same side or opposite sides of I according as I lies inside or outside the two circles. In the particular case when the transversal is a common tangent the corresponding and non-corresponding points coincide.

Cor. i.—A tangent drawn to one of two circles from either centre of similitude is also a tangent to the other circle, and the points of contact are the extremities of parallel radii.

Cor. ii.—The distances of two corresponding points from a centre of similitude have the same ratio as the radii of the circles.

Cor. iii.—Tangents at corresponding points are parallel.

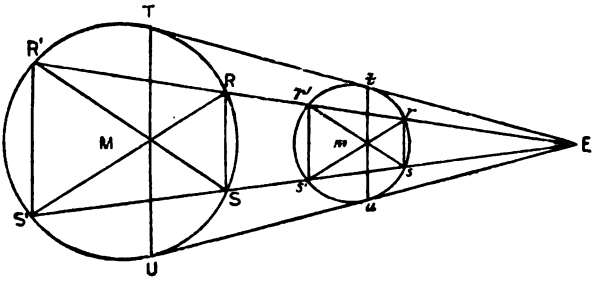
Proof.—The tangents are perpendicular to parallel radii.

(147) THEOREM.—*If two transversals be drawn to cut two circles through either centre of similitude, the chord joining any two points of section on one circle is parallel to the chord joining the corresponding points of section on the other circle.*

Proof.—Through E, one of the centres of similitude of the circles M and m, draw two transversals, one cutting circle M in R, R′ and circle m in r, r', where R, r and R′, r' are corresponding pairs, and the other cutting circle M in S, S′, and circle m in s, s' where S, s and S′, s' are corresponding pairs. Join RS, R′S′, RS′, R′S, rs, $r's'$, rs', $r's$.

Consider triangles ERS, E*rs*. By *Art.* 146, ii, ER : E*r* = radius of circle M : radius of circle *m* = ES : E*s*. And the angle at E is common. Therefore the triangles are similar. Therefore angles SRE, *s*rE are equal. Therefore RS is parallel to *rs*.

Similarly it may be proved that the following pairs of chords are parallel, R'S', *r's'*; RS', *rs'*; R'S, *r's*.



Cor. i.—If through either centre of similitude of two circles there be drawn a common tangent and a transversal, the chords joining the points of contact to either pair of corresponding points on the transversal are parallel.

Cor. ii.—If through either centre of similitude of two circles tangents be drawn to touch the circles, the chord of contact in one circle is parallel to the chord of contact in the other.

Cor. iii.—A transversal through either centre of similitude of two circles divides the circles into two pairs of similar segments.

Proof.—Since RS, R'S are parallel to *rs*, *r's* respectively, the angles RSR', *rsr'* are equal. Therefore the segments RSR', *rsr'* are similar.

The similar segments are on the same side of the transversal when it passes through the external centre of similitude, on opposite sides when it passes through the internal centre.

(148) THEOREM.—*If through either centre of similitude of two circles two transversals be drawn to cut the circles, any two points of section on one circle are concyclic with the non-corresponding points on the other circle.*

Proof.—In the figure of *Art.* 147, since RS, rs are parallel, angle R′RS = angle $r'rs$ = angle $r's'$S. Therefore angles r'RS, $r's'$S are supplementary (or equal). Therefore R, S, r', s' are concyclic. Similarly it may be proved that the following groups of points are concyclic :—R′, S′, r, s; R, S′, r', s; R′, S, r, s'.

Cor. i.—If through either centre of similitude of two circles there be drawn to the circles a common tangent and a transversal, the two points of contact are concyclic with either pair of non-corresponding points on the transversal.

Cor. ii.—If through either centre of similitude of two circles there be drawn two common tangents, the four points of contact are concyclic.

(149) THEOREM.—*If through either centre of similitude of two circles there be drawn to the circles any transversal, the rectangle contained by the distances of the centre of similitude from either pair of non-corresponding points is a constant.*

Proof.—In the figure of *Art.* 147, let a common tangent through E touch circle M at T and circle m at t. Then, by *Art.* 148, i, R, T, r', t are concyclic. Therefore ER.Er' = ET.Et, which is constant for the same centre of similitude. Similarly ER′.Er = ET.Et.

If the external centre of similitude E is taken, the constant of this article is positive when E is outside, negative when E is inside the circles; if the internal centre of similitude I is taken, the constant is negative when I is outside, positive when I is inside the circles. *Otherwise:* When the circles lie outside each other, the constant is positive for E, negative for I, when the circles intersect it is positive for both E and I, and when the circles lie one within the other, it is negative for E and positive for I.

Def.—The circle whose centre is a centre of similitude of two given circles, and whose radius is equal to the square root of the product of the distances of the centre of similitude from either pair of non-corresponding points on any transversal through the centre of similitude, is called a *circle of antisimilitude* of the two given circles. When the external centre of similitude is taken as centre, the circle is called the *external* circle of antisimilitude; when the internal centre of similitude is taken as centre, the circle is called the *internal* circle of antisimilitude.

When two circles lie outside each other, their external circle of antisimilitude is real, and their internal circle of antisimilitude imaginary.

When two circles intersect, both their circles of antisimilitude are real, the radius of either being the distance of its centre from one of the points of intersection.

When two circles lie one within the other, their external circle of antisimilitude is imaginary, and their internal circle of antisimilitude real.

(150) THEOREM.—*If through either centre of similitude of two circles, two transversals be drawn to the circles, the chord joining any two points of section on one circle, and the chord joining the two non-corresponding points of section on the other circle, meet on the radical axis of the two circles.*

Proof.—In the figure of *Art.* 147, RS is the common chord, and, therefore, the radical axis of the circle M and the circle which passes through RS$s'r'$. $r's'$ is the common chord, and, therefore, the radical axis of the circle m and the circle which passes through RS$s'r'$. Therefore by *Art.* 131, RS, $r's'$ and the radical axis of the circles M, m are concurrent.

Similarly the following pairs of chords intersect on the radical axis of the circles M and m:—R'S', rs; RS', $r's$; R'S, rs'.

Cor. 1.—If through either centre of similitude of two

circles there be drawn to the circles a common tangent and a transversal, the chords joining the points of contact to either pair of non-corresponding points on the transversal meet on the radical axis of the two circles.

Cor. ii.—If through either centre of similitude of two circles there be drawn to the circles two common tangents, the chords of contact meet on the radical axis of the circles (at infinity).

Cor. iii.—If through either centre of similitude of two circles a transversal be drawn to the circles, the tangents at either pair of non-corresponding points intersect on the radical axis of the two circles.

Proof.—RS, $r's'$ intersect on the radical axis, but RS, $r's'$ become the tangents at R, r' when ES is turned about E until it coincides with ER. *Otherwise:* The angles which the tangents at R, r' make with ER are equal, being equal to the angles in the segments RSR', rsr' respectively (*Art.* 147, iii). Therefore the tangents are equal and meet on the radical axis.

(151) THEOREM.—*The orthocentre of a triangle is the external, and the centroid the internal centre of similitude of its nine-point and circum-circles.*

Proof.—See the figure of *Art.* 22.

Let ρ be the radius of the nine-point circle, and R the radius of the circumcircle. Then $R = 2\rho$. But $HO = 2HN$. Therefore $HO : HN = R : \rho$. Therefore H is the external centre of similitude. Again $GO = -2GN$. Therefore $-GO : GN = R : \rho$. Therefore G is the internal centre of similitude.

(152) THEOREM.—*The vertices of a triangle, and the points in which the internal and external bisectors of its angles meet the opposite sides, are the centres of similitude of its inscribed and escribed circles taken in pairs.*

Proof.—See the figure of *Art.* 30.

Consider the circles I and I_1. Triangles AIY, AI_1Y_1 are similar. Therefore $AI : AI_1 = IY : I_1Y_1 = r : r_1$. Therefore A is the external centre of similitude. Again, triangles LIX, LI_1X_1 are similar. Therefore $LI : LI_1 = IX : I_1X_1 = -r : r_1$. Therefore L is the internal centre of similitude. Similarly for the inscribed circle and any other escribed circle.

Again, consider the circles I_1, I_2. The triangles $N'I_1Z_1$, $N'I_2Z_2$ are similar. Therefore $N'I_1 : N'I_2 = I_1Z_1 : I_2Z_2 = r_1 : r_2$. Therefore N' is the external centre of similitude. Again, the triangles CI_1Y_1, CI_2Y_2 are similar. Therefore $CI_1 : CI_2 = I_1Y_1 : I_2Y_2 = -r_1 : r_2$. Therefore C is the internal centre of similitude. Similarly for any other two escribed circles.

(153) THEOREM.—*The six centres of similitude of three circles taken in pairs form the vertices of a complete quadrilateral.*

Proof.—Let C, C', C'' be the centres of three circles whose radii are r, r', r'' respectively. Let I, I', I'' be the internal and E, E', E'' the external centres of similitude of the pairs of circles C, C'; C', C''; C'', C.

Then—

$$CE/EC' = -CI/IC' = -r/r';$$
$$C'E'/E'C'' = -C'I'/I'C'' = -r'/r'';$$
$$C''E''/E''C = -C''I''/I''C = -r''/r.$$

Therefore—

$$\frac{CE.C'E'.C''E''}{EC'.E'C''.E''C} = \frac{CI.C'I'.C''E''}{IC'.I'C''.E''C}$$
$$= \frac{C'I'.C''I''.CE}{I'C''.I''C.EC'} = \frac{C''I''.CI.C'E'}{I''C.IC'.E'C''} = -\frac{r.r'.r''}{r.r'.r''} = -1.$$

Therefore, by *Art.* 17, the following are triads of collinear points:—E, E', E''; I, I', E''; I', I'', E; I'', I, E'.

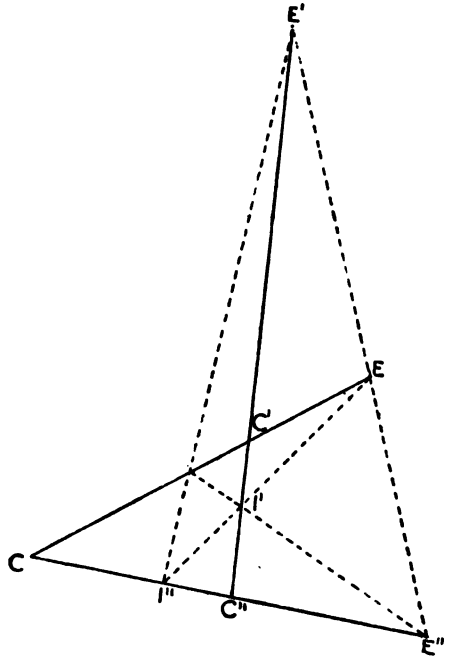
Cor. i.—The triangle formed by the lines of centres of three circles is the diagonal triangle of the complete quadrilateral, whose vertices are the six centres of similitude of the circles taken in pairs.

Cor. ii.—The connectors of the centre of each of three circles, with the internal centre of similitude of the other two, are concurrent.

Proof.—

$$\frac{CI.C'I'.C''I''}{IC'.I'C''.I''C} = \frac{r.r'.r''}{r.r'.r''} = 1.$$

Therefore, by *Art.* 16, C''I, CI', C'I'' are concurrent.



Defs.—The four straight lines on which lie the six centres of similitude of three circles taken in pairs are called the *axes of similitude* of the circles. The straight line on which lie the three external centres of similitude is

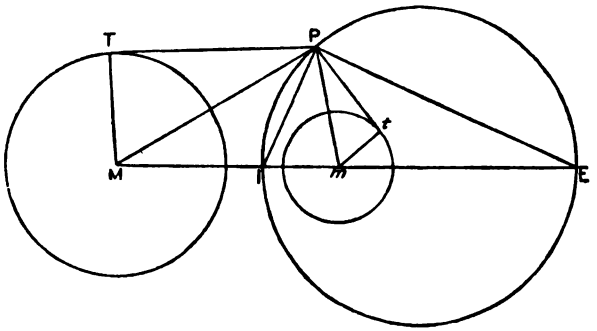
called the *external axis of similitude.* The three others are called the *internal axes of similitude.*

(154) *Def.*—The circle described on the distance between the two centres of similitude of two circles as diameter, is called the *circle of similitude* of the circles.

THEOREM.—*If* r *is the radius and* C *the centre of the circle of similitude of two circles whose centres are* C′, C″, *then* $r^2 = CC'.CC''$.

Proof.—Let the circle C cut C′C″ in I and E, the internal and external centres of similitude respectively. Then EIC′C″ is a harmonic range, and C is the middle point of EI. Therefore $CI^2 = CC'.CC''$. Therefore $r^2 = CC'.CC''$.

(155) THEOREM.—*Tangents drawn to two circles from any point on their circle of similitude have the same ratio as their radii.*



Proof.—Let I, E be the internal and external centres of similitude respectively of two circles whose centres are *M* and *m*. From P any point on the circle of similitude, let tangents PT, P*t* be drawn to the circles M and *m* respectively. Join MT, *mt*, PM, P*m*, PI, PE. Then since M*m* IE is a harmonic range, P.M*m*IE is a harmonic pencil.

But IPE is a right angle. Therefore, PI bisects the angle MP*m*. Therefore, signs being disregarded, PM : P*m* = MI : *m*I. But MI : *m*I = MT : *mt*. Therefore PM : P*m* = MT : *mt*. And the angles at T, *t* are right. Therefore, by *Euc.* vi. 7, the triangles PMT, P*mt* are similar. Therefore PT : P*t* = MT : *mt*.

Cor. i.—If PX'Y', PX''Y'' are two secants through P to the circles M, *m* respectively, then PX'.PY' and PX''.PY'' have the same ratio as the square of the radii.

Cor. ii.—Circles subtend equal angles at any point on their circle of similitude.

Proof.—The angles MPT, *m*P*t* are equal. Therefore twice MPT equals twice *m*P*t*.

Cor. iii.—Two circles and their circle of similitude are coaxal.

Proof.—PT : P*t* = MT : *mt* = a constant. Therefore, by *Art.* 141 *Converse*, the locus of P is a circle coaxal with the circles M and *m*.

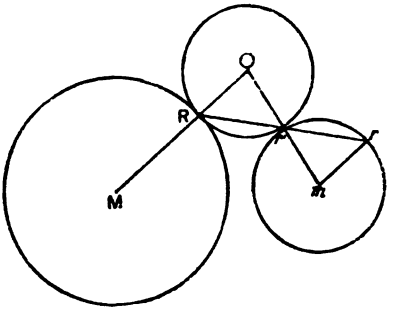
(156) THEOREM.—*The three circles of similitude of three given circles taken in pairs are coaxal.*

Proof.—See the figure of *Art.* 153.

The circles of similitude of the circles C, C', C'' are the circles described on IE, I'E', I''E'', the three diagonals of the complete quadrilateral EE'II'I''E'' as diameters. Therefore, by *Art.* 143, they are coaxal.

(157) THEOREM.—*If a variable circle touches two fixed circles, the connector of the points of contact passes through a centre of similitude of the fixed circles.*

Proof.—Let the variable circle O touch the fixed circles M and *m* in R and *r'* respectively. Join MR, *mr'*, R*r'*. Let R*r'* meet circle *m* again in *r*. Join *mr*. MR, *mr'* will intersect in O. Then angle OR*r'* = angle O*r'*R = angle *mr'r* = angle *mrr'*. Therefore MR, *mr* are parallel radii. Therefore R*r'* passes through a centre of similitude of circles M and *m*.



Cor. i.—The centre of similitude is external or internal according as the contacts of the variable circle with the fixed circles are of the same kind or of different kinds.

The direct common tangents are particular cases of the variable circle having contacts of the same kind; the transverse common tangents are particular cases of the variable circle having contacts of different kinds.

The student should draw the various possible figures for himself, including the cases where one of the fixed circles lies within the other.

Cor. ii.—The rectangle contained by the segments of a secant drawn to the variable circle from the centre of similitude is constant.

Proof.—Let E be the centre of similitude, EXY the secant. Then $EX.EY = ER.Er' = a$ constant.

Cor. iii.—The tangent from the centre of similitude to the variable circle is constant.

Cor. iv.—The radical axes of every two of the circles obtained by taking different positions of the variable circle pass through the centre of similitude.

Cor. v.—The centre of similitude is the radical centre of every three of the circles obtained by taking different positions of the variable circle.

Cor. vi.—One of the centres of similitude of the circles

obtained by taking any two positions of the variable circle lies on the radical axis of the fixed circles.

The centre of similitude is external or internal according as one (and consequently the other) of the fixed circles has or has not contacts of the same kind with the two positions of the variable circle.

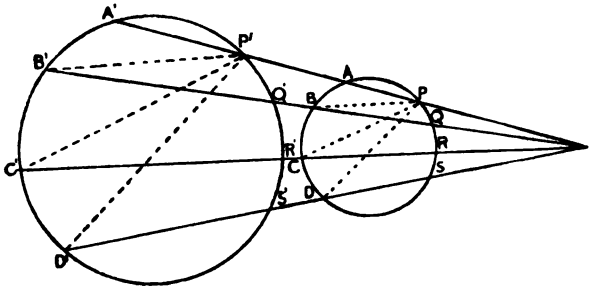
This is simply *Cor.* iv. with the fixed and variable circles interchanged.

Cor. vii.—The radical axis of the fixed circles is an axis of similitude of the circles obtained by taking any three positions of the variable circle.

The axis of similitude is external or internal according as one (and consequently the other) of the fixed circles has or has not contacts of the same kind with the three positions of the variable circle.

This follows immediately from *Cor.* vi.

(158) THEOREM.—*Transversals through either centre of similitude of two circles cut the circles homographically.*



Proof.—Let the transversals cut one of the circles in the pairs of coradial points PA, QB, RC, SD . . ., and on the other circle let P′Q′R′S′ . . ., A′B′C′D′ . . . correspond to PQRS . . ., ABCD . . . respectively. Join P to B, C, D, &c., and P′ to B′, C′, D′, &c. Then, by *Art.* 147 the follow-

ing pairs of lines are parallel: PB, P'B'; PC, P'C'; PD, P'D', &c. Therefore P.ABCD . . ., P'.A'B'C'D' . . . are equiangular, and consequently homographic. Therefore (ABCD . . .) = (A'B'C'D' . . .). But by *Art.* 112 (ABCD . . .) = (PQRS . . .), and (A'B'C'D' . . .) = (P'Q'R'S' . . .). Therefore (PQRS . . .) = (ABCD . . .) = (P'Q'R'S' . . .) = (A'B'C'D' . . .).

CHAPTER XII.

INVERSION.

(159) *Defs.*—If O be a fixed point, P any other point, and P′ a point collinear with O, P, such that $OP.OP' = k$, a constant, P and P′ are said to be *inverse points* with reference to the circle whose centre is O and whose radius $= \sqrt{k}$. This circle is termed the *circle of inversion*, its centre O the *centre of inversion*, and the constant k, the *constant of inversion*. Any figure traced out by either of the points P, P′ is said to be the *inverse* of the figure traced out by the other point.

The circle of inversion is real when k is positive, *i.e.* when P and P′ lie on the same side of O; it is imaginary when k is negative, *i.e.* when P and P′ lie on opposite sides of O. When $k = 0$, the circle of inversion coincides with O, and O and every point in the plane are inverse points; when k does not equal 0, O and every point at infinity in the plane are inverse points. Obviously every straight line through the centre of inversion cuts two inverse figures in two inverse points (which may coincide), and a point of intersection of two inverse figures is its own inverse.

(160) THEOREM.—*If* ABCD *is a harmonic range*, C *and* D *are inverse points with respect to the circle described on* AB *as diameter.*

Proof.—Let M be the middle point of AB. Then $MC.MD = MA^2$. Therefore C and D are inverse points

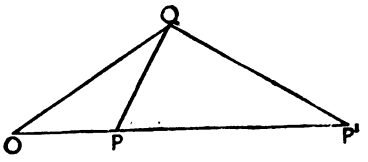
with respect to the circle whose centre is M and radius MA.

Converse.—The line joining two inverse points is divided harmonically by the circle of inversion.

(161) THEOREM.—*Any circle through two inverse points cuts the circle of inversion orthogonally.*

This follows at once from the above converse, and *Art.* 125, *Converse.*

(162) THEOREM.—*If* O *be the centre of inversion*, P, P *two inverse points, and* Q *any point on the circle of inversion, then* $PQ^2 : P'Q^2 = OP : OP'$.



Proof.—Join OQ. Since OQ is a radius of the circle of inversion, $OQ^2 = OP.OP'$. Therefore $OP : OQ = OQ : OP'$. Therefore, and since the angle at O is common, the triangles OPQ, OQP' are similar. Therefore (*Euc.* vi. 19) they have to each other the ratio $PQ^2 : P'Q^2$. But (*Euc.* vi. 1) they have to each other the ratio $OP : OP'$. Therefore $PQ^2 : P'Q^2 = OP : OP'$.

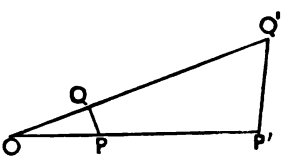
(163) THEOREM.—*Two points and their inverses are concyclic.*

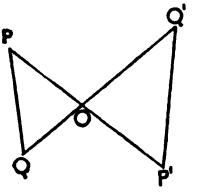
Proof.—Let O be the centre of inversion, P, Q the two points, P', Q' their respective inverses. Then $OP.OP' = OQ.OQ'$. Therefore (*Euc.* iii. 35, 36) P, P', Q, Q', are concyclic.

Converse.—A circle cutting two inverse figures in two inverse points cuts them in other two inverse points.

Cor.—If a circle passing through two inverse points touches one of two inverse figures at one of the points, it touches the other figure at the other point.

Proof.—When P, Q coincide, P′, Q′ coincide. Therefore if the circle round P, P′, Q, Q′ meets one of the inverse figures in two coincident points, it meets the other in two coincident points, *i.e.* if it touches one figure at P, it touches the other at P′.





(164) THEOREM.—*A straight line joining two inverse points cuts the inverse figures to which they belong at equal angles*, i.e. *the tangents to the figures at these points are equally inclined to the line.*

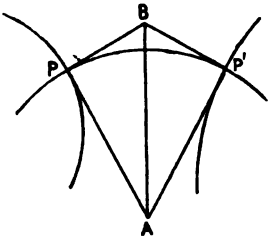
Proof.—In the figure of *Art.* 163, join PQ, P′Q′. Then by *Euc.* iii. 21, 22, PQ makes the same angle with OPP′ that P′Q′ makes with OQQ′. This, being true universally, is true in the case where OQQ′ coincides with OPP′. But in this case PQ and P′Q′ are the tangents to the inverse figures at P and P′. Therefore these tangents are equally inclined to OPP′.

Cor.—The tangents to two inverse figures at two inverse points are equal (*Euc.* i. 6).

(165) THEOREM.—*A circle through two inverse points cuts the inverse figures to which the points belong at equal angles*, i.e. *the tangents to the circle and the figures at these points are equally inclined to each other.*

Proof.—Let the tangents to the inverse figures at the

inverse points P, P′ meet in A, and let the tangents to the circle through P, P′, at these points meet in B. Join AB.



Then AB = AB, PB = P′B, and by *Art.* 164, *Cor.*, PA = P′A. Therefore angle APB = angle AP′B.

Hence might be deduced *Art.* 163, *Cor.*

(166) THEOREM.—*If the circle of inversion cuts a figure at any point, the figure is cut by its inverse at the same point.*

Proof.—Let O be the centre of inversion. Let the circle of inversion cut the figure at P. Then OP is a radius of the circle of inversion. Let OP cut the inverse figure at P′. Then $OP.OP' = OP^2$. Therefore OP = OP′. Therefore P′ coincides with P.

Converse.—If a figure and its inverse cut at a point, the circle of inversion cuts both at that point.

Proof.—Let the two figures meet at P. Then P is its own inverse. Therefore the square of the radius of the circle of inversion = OP.OP. Therefore OP = radius of the circle of inversion. Therefore the circle of inversion passes through P.

Cor. i.—When the circle of inversion cuts two inverse figures, it cuts them at equal angles.

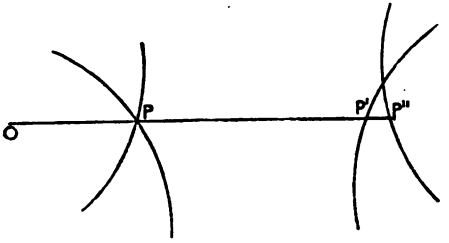
Proof.—The tangents to the two figures at P make equal angles with OP (*Art.* 164), and therefore with the tangent to the circle of inversion at P, which is perpendicular to OP.

Cor. ii.—If a figure, its inverse, and the circle of inversion meet at a point, and two of them touch at it, all three touch at it.

Proof.—The tangent to the circle of inversion at the point makes equal angles with the tangents to the figures at the point. Therefore if two of the tangents coincide, all three coincide.

Cor. iii.—If two inverse figures meet, the circle of inversion is real.

(167) THEOREM.—*When two figures meet at a point, their inverses meet at the inverse of that point.*

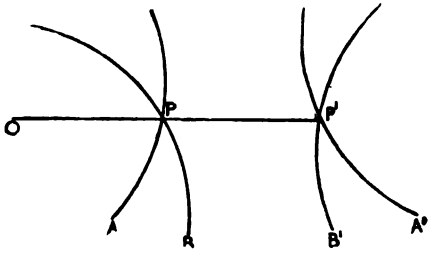


Proof.—Let O be the centre of inversion, and let the two figures meet at P. Let the connector OP meet one of the inverse figures in P′ and the other in P″. Then P′ and P″ are both inverses of P. Therefore OP.OP′ = OP.OP″. Therefore OP′ = OP″. Therefore P″ coincides with P′.

(168) THEOREM.—*Two figures intersect at the same angle as their inverses.*

Proof.—Let O be the centre of inversion, A and B two figures intersecting at P, A′ and B′ their respective inverses intersecting at P′. Join O, P, P′ which are collinear. Then (*Art.* 164) the angle between OPP′ and A = angle between OPP′ and A′; and the angle between OPP′ and B = angle

between OPP′ and B′. By addition or subtraction we obtain that the angle between A and B = the angle between A′ and B′.



Cor.—If two figures touch at a point, their inverses touch at the inverse of that point.

Proof.—Let A and B touch at P. Then the angle between them = 0. A and B′ meet at the inverse of P, and the angle between them = angle between A and B, *i.e.* 0. Therefore A′ and B′ touch at the inverse of P.

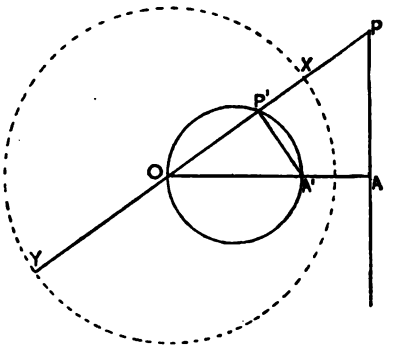
(169) THEOREM.—*The inverse of a straight line is a circle through the centre of inversion, and such that the diameter terminated by this centre is perpendicular to the straight line.*

Proof.—From O, the centre of inversion, draw OA perpendicular to the given line. Let A′ be the inverse of A. Take any point P in the given line. Join OP. Let P′ be the inverse of P. Join A′P′. Then A, P, P′, A are concyclic, and the angle at A is a right angle. Therefore the angle at P′ is a right angle. Therefore the locus of P′ is a circle having OA′ as diameter.

Cor.—A straight line is the radical axis of its inverse and the circle of inversion.

Proof.—Let the circle of inversion cut OP in X and Y. Then $PX.PY = OP^2 - OX^2 = OP^2 - OP.OP' = OP(OP - OP')$

$= PO.PP'$. Therefore P is a point on the radical axis of the circles, and PA is perpendicular to their line of centres. Therefore it is the radical axis.



Particular case.—The inverse of a straight line passing through the centre of inversion is the straight line itself.

(170) THEOREM (converse of last).—*The inverse of a circle through the centre of inversion is a straight line perpendicular to the diameter through the centre of inversion.*

Proof.—Through O, the centre of inversion, draw CA' a diameter to the given circle. Let A be the inverse of A'. Take any point P' on the circle. Join OP'. Let P be the inverse of P'. Join A'P' and AP. Then since A, P, P', A' are concyclic and the angle at P' is right, being the angle in a semicircle, the angle at A is right. Therefore, since A is a fixed point, the locus of P is a straight line perpendicular to OA.

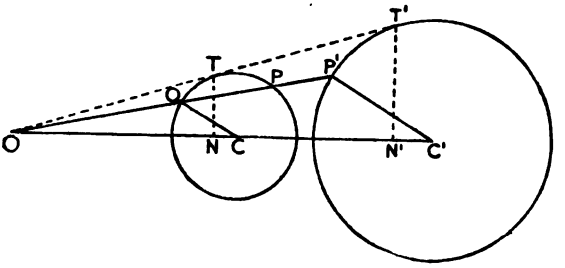
Particular case.—Since O and any point at infinity are inverse points, the inverse of O, regarded as a point-circle, is the straight line at infinity.

(171) THEOREM.—*The inverse of a circle which does not pass through the centre of inversion is a circle whose centre is*

collinear with the centre of inversion and the centre of the original circle.

Proof.—Let O be the centre of inversion, C the centre of the given circle, P any point on that circle. Join OP. Let OP cut the circle again in Q, and let P′ be the inverse of P. Join QC and OC. Through P′ draw a parallel to QC meeting OC in C′.

Then OP.OP′ = a constant, and OP.OQ = a constant. Therefore OP′ : OQ = a constant. But OC′ : OC = OP′ : OQ. Therefore OC′ : OC = a constant. But O, C are fixed points. Therefore C′ is a fixed point. Again, C′P′ : CQ = OP′ : OQ. Therefore C′P′ : CQ = a constant. But CQ = a constant.



Therefore C′P′ = a constant. And C′ is a fixed point. Therefore the locus of P′ is a circle having C′ as centre.

Cor. i.—O is a centre of similitude of the two circles.

Proof.—OC′ : OC = C′P′ : CQ. (See *Art.* 144, *Def.*)

Cor. ii.—If the centre of inversion lies inside a circle, it lies inside the inverse circle; if it lies outside a circle, it lies outside the inverse circle (*Art.* 144).

Cor. iii.—Inverse points on the circles are antihomologous with respect to the centre of inversion. (See *Art.* 146, *Def.*)

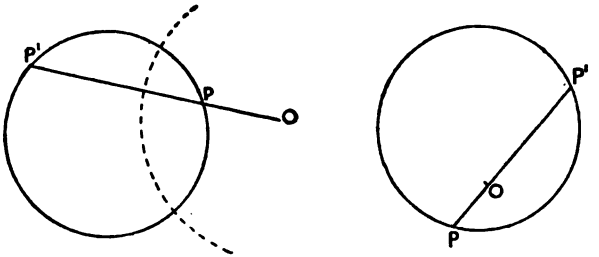
Cor. iv.—The circle of inversion is a circle of antisimilitude of the two inverse circles. (See *Art.* 149, *Def.*)

Cor. v.—Any two circles are inverses with respect to either centre of similitude as centre of inversion and the

product of the distances of any two antihomologous points from the centre of similitude as constant of inversion.

It should be observed that C and C′ are not inverse points. If through O a common tangent be drawn touching the circles C, C′ in T, T′ respectively, and if N, N′ be the feet of perpendiculars on CC′ from T, T′ respectively, then C, N′ and C′, N are pairs of inverse points.

(172) THEOREM (particular case of last).—*A circle is its own inverse with respect to an orthogonal circle, real or imaginary.*



Proof.—Through O, the centre of inversion, draw OPP′ any secant to the given circle. Then, since the circle of inversion is orthogonal, OP.OP′ = square of radius of circle of inversion = constant of inversion. Therefore P, P′ are inverse points. But as P traces out one part of the circle, P′ traces out the other. Therefore the circle is its own inverse.

When O is inside the given circle the circle of inversion is imaginary.

The theorem may be stated otherwise, thus:—A circle is its own inverse when the constant of inversion is equal to the product of the distances of the centre of inversion from the extremities of a chord through the centre of inversion.

Cor.—Two (or three) circles can be inverted into them-

selves by taking a (or the) circle orthogonal to them as the circle of inversion.

(173) THEOREMS.—

(i) *A circle and a diameter invert in general into two orthogonal circles* (*Art.* 168). *If the centre of inversion lies on the circle, a circle and a diameter invert into a straight line and a circle having the straight line as diameter.*

(ii) *A circle and a tangent invert in general into two circles which touch each other* (*Art.* 168, *Cor.*). *If the centre of inversion lies on the circle, a circle and a tangent invert into a straight line and a circle touching it.*

(iii) *Parallel straight lines invert into circles which touch each other.*

Proof.—The inverses of the straight lines are circles which meet in the centre of inversion (*Art.* 169). These circles make with each other the same angle as the straight lines (*Art.* 168), viz., zero. Therefore they touch at the centre of inversion.

Converse.—If the point of contact be taken as the centre of inversion, circles which touch each other invert into parallel straight lines.

Cor.—The line of centres of the circles is perpendicular to the straight lines.

(iv) *Circles which touch each other invert, in general, into circles which touch each other.*

(v) *A pencil of rays inverts into a system of common-point coaxal circles.*

Proof.—The rays invert into circles which pass through the centre of inversion (*Art.* 169), and which also pass through the point inverse to the point of intersection of the rays (*Art.* 167).

Converse.—If one of the common points is taken as

the centre of inversion, a system of common-point coaxal circles inverts into a pencil of rays.

Cor. i.—The two common points and the vertex of the pencil are collinear.

Cor. ii.—The pencil is homographic with the pencil formed by the tangents to the circles at a common point.

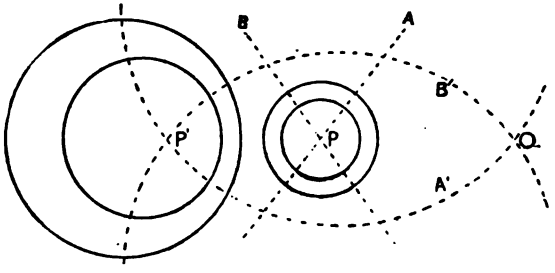
Proof.—The two pencils are equiangular (*Art.* 168).

(vi) *Common-point coaxal circles invert in general into common-point coaxal circles.*

Cor.—The pencils formed by the tangents at the common points are homographic.

(vii) *Concentric circles, when the common centre is taken as the centre of inversion, invert into concentric circles with the same centre.*

(viii) *Concentric circles invert in general into limiting-point coaxal circles.*



Proof.—Let O be the centre of inversion, P the centre of the concentric circles, P′ the inverse of P. Through P draw any two straight lines A and B. Their inverses, A′ and B′, are circles through O and P′ (*Th.* v). Since A and B cut the concentric circles orthogonally, A′ and B′ cut the circles inverse to the concentric circles orthogonally (*Art*

168). Therefore these circles are coaxal (*Art.* 135, viii). They cannot intersect because their inverses do not intersect. Therefore they are limiting-point coaxal circles.

Cor. 1.—O and P′ are the limiting points.

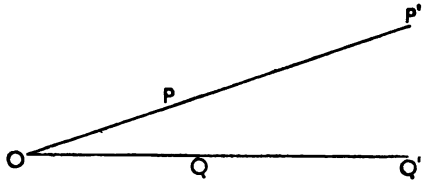
Converse.—Limiting-point coaxal circles, when one of the limiting points is taken as the centre of inversion, invert into concentric circles which have the inverse of the other limiting point as centre.

Cor. 2.—The common radical axis of the limiting-point circles is the inverse of that one of the concentric circles which passes through the centre of inversion.

Cor. 3.—The inverses of those concentric circles to which O is external have O lying outside them (*Art.* 171, ii), *i.e.* they lie on the opposite side of their common radical axis from O; the inverses of those concentric circles to which O is internal have O lying inside them (*Art.* 171, ii), *i.e.* they lie on the same side of their common radical axis as O.

Cor. 4.—Any two non-intersecting circles can be inverted into two concentric circles by taking a limiting point as centre of inversion.

(ix) *Limiting-point coaxal circles invert in general into limiting-point coaxal circles.*

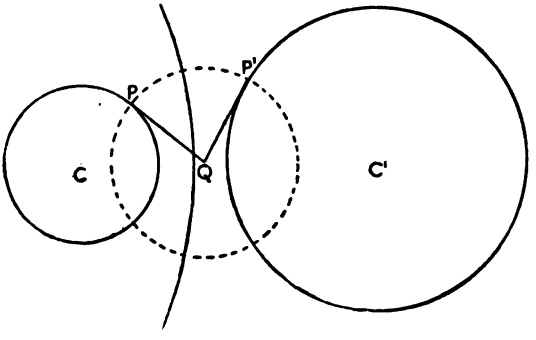


Proof.—Let O be the centre of inversion; P, Q the limiting points of a given system of coaxal circles A; P′, Q′ the inverses of P, Q; and A′

the circles inverse to A. Then circles through P, Q invert into circles through P′, Q′ (*Th.* vi). But circles through P, Q cut the circles A orthogonally (*Art.* 135, v), therefore circles through P′, Q′ cut the circles A′ orthogonally (*Art.* 168). Therefore the circles A′ are coaxal and have P′, Q′ as their limiting points (*Art.* 135, x).

Cor.—The radical axis of either system is the inverse of that circle of the other system which passes through the centre of inversion.

(174) THEOREM.—*A circle, its inverse, and the circle of inversion are coaxal.*

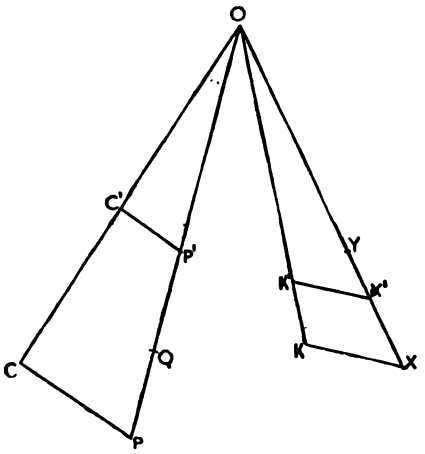


Proof.—Let O, C, C′ be the centres of the circle of inversion, and of two inverse circles respectively, and let P, P′ be two inverse points on the circles C, C′. Draw tangents at P, P′ to the circles C, C′, meeting in Q. Then since QP = QP′ (*Art.* 164, *Cor.*), the circle described with centre Q and radius QP passes through P, P′. Therefore it cuts the circle of inversion orthogonally (*Art.* 161). It also cuts the circles C, C′ orthogonally (*Art.* 124). Further O, C, C′ are collinear. Therefore (*Art.* 135, ix) the circles O, C, C′ are coaxal.

Cor. i.—Two circles and a circle of antisimilitude are coaxal (*Art.* 171, iv).

Cor. ii.—If from a point A on the circle of inversion (or antisimilitude) secants AXY, AX'Y' and tangents AT, AT' are drawn to the circles C, C' respectively, then $AX.AY : AX'.AY' = AT^2 : AT'^2 = OC : OC'$ = radius of C : radius of C' (*Arts.* 141 and 144).

(175) PROBLEM.—*To construct two circles which shall be the inverses of two given circles and equal to each other.*



Construction.—On a (real) circle of antisimilitude of the circles C, K, take any point O. From O draw the secants OPQ, OXY to the circles C, K respectively. Take any point P' on OP. From OX cut off OX' such that OP'.OQ = OX'.OY. Through P', X' draw parallels to PC, XK respectively meeting OC, OK in C', K' respectively. With centres C', K' and radii C'P', K'X' respectively describe two circles. These are the circles required.

Proof.—O is obviously a centre of similitude of the

circles C, C′ and of the circles K, K′. Further OP′.OQ = OX′.OY. Therefore circles C′, K′ are the inverses of circles C, K with respect to O as the centre of inversion.

Again $\frac{OP.OQ}{OP'.OQ} = \frac{OP}{OP'} = \frac{CP}{C'P'}$; and $\frac{OX.OY}{OX'.OY} = \frac{OX}{OX'} = \frac{KX}{K'X'}$.

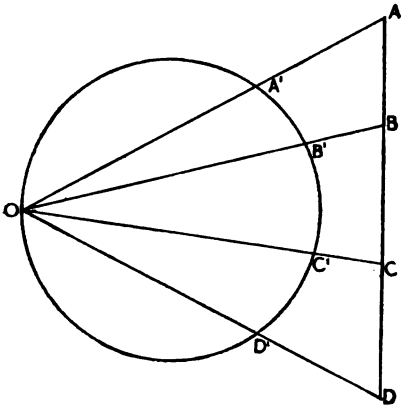
But OP′.OQ = OX′.OY. Therefore, by division,

$\frac{OP.OQ}{OX.OY} = \frac{CP.K'X'}{C'P'.KX}$. But $\frac{OP.OQ}{OX.OY} = \frac{CP}{KX}$ (*Art.* 174, *Cor.* ii).

Therefore $\frac{CP.K'X'}{C'P'.KX} = \frac{CP}{KX}$. Therefore C′P′ = K′X′. Therefore the circles C′, K′ are equal.

Cor.—Three circles can be inverted into three equal circles by taking a point of intersection of the circles of antisimilitude of the circles taken in pairs as the centre of inversion.

(176) THEOREM.—*Points on a straight line or circle are homographic with their inverse points.*

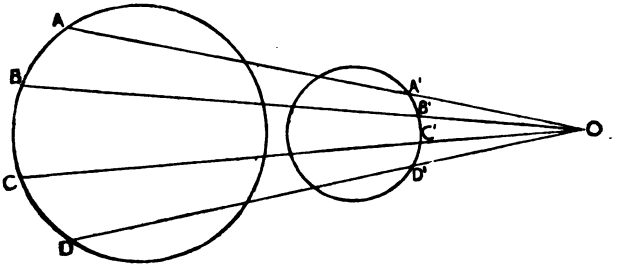


Proof.—

(i) When A, B, C, D . . . lie on a straight line, and their inverses A′, B′, C′, D′ . . . lie on a circle, the connectors AA′, BB′, CC′, DD′ . . . intersect

on the circle in O, the centre of inversion. And O (A'B'C'D' . . .) = O (ABCD . . .). Therefore (A'B'C'D' . . .) = (ABCD . . .).

(ii) When A, B, C, D . . . and their inverses A', B', C', D' . . . lie on two inverse circles, AA' BB'



CC', DD' . . . intersect in O, a centre of similitude of the circles. Therefore, by Art. 158 (ABCD . . .) = (A'B'C'D' . . .).

(iii) When the points and their inverses lie on a straight line through O inverse to itself, they form an involution, and the theorem is a particular case of *Art.* 117.

(iv) When the points and their inverses lie on the same circle inverse to itself, the theorem is a particular case of *Art.* 112.

(177) *Inversion.*—By transforming the elements in a given theorem, in accordance with the results of the preceding articles, we can obtain from it a corresponding or inverse theorem. This process, which supplies us with a means of passing from known and simple propositions to others of a more general character, and which is similar in some respects to that of reciprocation, is termed *Inversion.* All theorems which confine themselves to considerations of the contact, intersection, angles of intersection, coaxality, and homographic division of straight lines and circles (including therefore all theorems capable of reciprocation

in the sense of *Art.* 75) are capable of inversion. The following are a few examples of the use of the method. The inverse theorems are given opposite to each other in parallel columns, the theorems which are regarded as derivative occupying the right-hand column. The student should draw the figures and try to effect the inversion for himself before looking at this column.

If A, B, P, Q are four points on a circle, the connectors AP, BP, AQ, BQ make equal or supplementary angles at P and Q (*Euc.* iii. 21, 22).

If A, B, P, Q are four points on a circle (or straight line), and O any other point, the circles OAP, OBP, OAQ, OBQ make equal or supplementary angles at P and Q.

If A, B, P are three points on a circle, the radii through A and B meet at an angle which is double of the angle between the connectors AP and BP (*Euc.* iii. 20).

If A, B, P are three points on a circle (or straight line), and O any other point, the circles through O, A and O, B orthogonal to the given circle (or straight line) meet at an angle which is double of the angle between the circles OAP and OBP.

The altitudes of a triangle are concurrent.

If through a point O three circles be drawn intersecting two and two in A, B, C, the three circles drawn through O and A, O and B, O and C orthogonal to the circles OBC, OCA, OAB respectively have another common point besides O.

If through a fixed point chords AA′, BB′, CC′, &c., are drawn to a given circle, the pairs of connectors AB, A′B′; AB′, A′B; AC, A′C′;

If through a point O and another fixed point circles are drawn cutting a given circle in the pairs of points A, A′; B, B′; C, C′, &c.,

AC', A'C, &c., intersect on a fixed straight line (*Art.* 87, i).

A variable tangent to a fixed circle cuts two fixed tangents in two homographic ranges (*Art.* 111).

A variable straight line touching one of a system of concentric circles cuts every other circle of the system which it does cut at a constant angle.

Two tangents to a given circle are equally inclined to the diameter passing through their point of intersection.

The three radical axes of three circles taken in pairs are concurrent.

the pairs of circles OAB, OA'B'; OAB', OA'B; OAC, OA'C'; OAC', OA'C, &c., intersect on a fixed circle through O.

A variable circle passing through a fixed point and touching a fixed circle, cuts two fixed circles passing through the same point and touching the same circle, in two series of points which are homographic.

A variable circle passing through a fixed point and touching one of a system of limiting-point coaxal circles cuts every other circle of the system which it does cut at a constant angle.

Two intersecting circles which touch a given circle are equally inclined to the circle passing through their points of intersection and orthogonal to the given circle.

If P, Q, R are three circles, and L, M, N other three circles passing through a point O, and such that L is coaxal with P, Q; M with Q, R; and N with R, P, then L, M, N have another common point besides O.

A variable straight line intersecting two fixed circles and passing through either centre of similitude (*i.e.* a variable straight line intersecting two fixed circles at equal or supplementary angles) cuts the circles in four homographic series of points (*Art.* 158).

A variable circle passing through a fixed point and intersecting two fixed circles at equal or supplementary angles cuts the circles in four homographic series of points.

(178).—By taking the centre of inversion in particular positions we can obtain particular cases of inverse theorems. For example:—

Two fixed straight lines u, u' are cut by straight lines passing through a fixed point P in two homographic ranges (*Art.* 99).

Two fixed straight lines u, u' intersecting in O are cut by circles passing through O and another fixed point P in two homographic ranges.

In this example the point of intersection of u, u' has been selected as the centre of inversion, and u, u' consequently invert into themselves (*Art.* 169).

The extremities of a diagonal of a complete quadrilateral are harmonic conjugates with respect to the points where it is cut by the other diagonals (*Art.* 80).

Let u, u', u'' be three straight lines intersecting in O. Let P be any other point on u. Through O, P draw any two circles cutting u' in A, C and u'' in B, D. Through O, A, D and O, C, B draw two circles cutting u in Q and R respectively. Then P bisects QR (*Art.* 57, ii).

In this example one of the vertices of the quadrilateral has been selected as the centre of inversion.

(179).—It will be seen from the inversion of the following two simple figures, to which others might easily be added, that the method of inversion may be applied to theorems involving consideration of the magnitude of straight lines.

(i) *To invert the straight lines* AB, AC, AD . . . *which have a common point* A, *and which are equal to each other.* The lines are the radii of a circle, *i.e.* they meet at a point and cut a certain circle orthogonally. Consequently their inverses A'B', A'C', A'D' . . . are arcs of circles which have two common points, viz., the centre of inversion and A' the inverse of A, and which cut a circle orthogonally at B', C', D' . . ., the inverses of B, C, D . . . .

(ii) *To invert the segment* PQ *bisected at* R. R is the point of intersection of PQ with the straight line which is perpendicular to PQ and also a diameter of any circle through P and Q. The points P, Q invert into points P', Q'; the straight line PQ inverts into the circle through P', Q' and O, the centre of inversion; and the middle point R inverts into R' the point of intersection of the circle OP'Q' with the circle through O orthogonal to the circle OP'Q' and to any other circle through P' and Q'. Since the circle orthogonal to the circle OP'Q', and to any other circle through P' and Q', is orthogonal to all circles through P' and Q' (*Art.* 135, vi), R' is the point of intersection of the circle OP'Q' with the circle through O orthogonal to all circles through P' and Q'.

The following are some examples of the application of these principles:—

(i) The angles at the base of an isosceles triangle are equal.

Inverse.—If three circles through a point O intersect

two and two in the points A, B, C; and if a circle can be drawn through B, C orthogonal to the circles OBA, OCA; then the circles OBA, OBC cut at the same angle as the circles OCA, OBC.

(ii) The diagonals of a rhombus cut each other at right angles.

Inverse.—If OBA, OCD are two circles touching each other at O, and ODA, OBC other two circles also touching each other at O, these four circles intersecting in A, B, C, D, and if a circle can be drawn through B, D orthogonal to the circles OBA, ODA, then the circles about OBD and OAC are orthogonal to each other.

(iii) The medians of a triangle are concurrent.

Inverse.—If three circles through a point O intersect two and two in the three points A, B, C; and if A′ is the point of intersection of the circle OBC with the circle orthogonal to all circles through B, C; and B′ the point of intersection of the circle OCA with the circle orthogonal to all circles through C, A; and C′ the point of intersection of the circle OAB with the circle orthogonal to all circles through A, B; the three circles through O, A, A′ and O, B, B′ and O, C, C′ have another common point besides O.

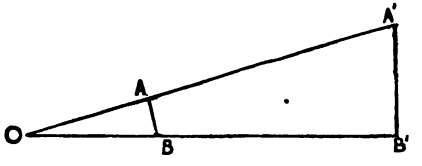
(iv) The diagonals of a parallelogram bisect each other.

Inverse.—If OBA, OCD are two circles touching each other at O, and ODA, OBC other two circles also touching each other at O, these four circles intersecting in A, B, C, D, and if the two circles through O, A, C and O, B, D intersect in E, then the circle through O, E orthogonal to the circle OAC is orthogonal to all circles through A, C, and the circle through O, E orthogonal to the circle OBD is orthogonal to all circles through B, D.

The theorems which follow, (*Arts.* 180 to 182), render possible a still further extension of the method of inversion to theorems involving the magnitude of straight lines.

(180) THEOREM.—*If* A, A′ *and* B, B′ *are two pairs of inverse points*, O *the centre and* k *the constant of inversion, then*

$$A'B' : AB = k/OA.OB.$$



Proof.—Since A, B, B′, A′ are concyclic, the triangles OAB, OB′A′ are similar. Therefore $A'B'/AB = OB'/OA = OB'.OB/OA.OB = k/OA.OB$.

Cor. i.—$AB/A'B' = k/OA'.OB'$.

Cor. ii.—If A, A′ and B, B′ and C, C′ are three pairs of inverse points, then

$$\frac{B'C'}{BC.OA} = \frac{C'A'}{CA.OB} = \frac{A'B'}{AB.OC}.$$

Proof.—By the theorem each of the above ratios $= k/OA.OB.OC$.

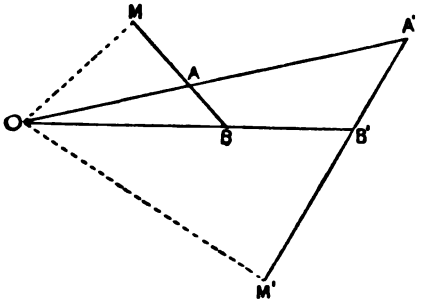
Cor. iii.—If A, A′; B, B′; C, C′; and D, D′ are four pairs of inverse points, then

$$\frac{A'B'.C'D'}{AB.CD} = \frac{B'C'.A'D'}{BC.AD} = \frac{C'A'.B'D'}{CA.BD}.$$

Proof.—By the theorem each of the above ratios $= k^2/OA.OB.OC.OD$.

(181) THEOREM.—*If* A, A′ *and* B, B′ *are two pairs of inverse points*, O *the centre of inversion and* OM, OM′ *perpendiculars on the connectors* AB, A′B′ *respectively, then* $AB/A'B' = OM/OM'$.

Proof.—Since the angles OAM, OB′M′ are equal, and the angles OMA, OM′B′ are equal, the triangles OAM, OB′M′



are similar. Therefore $OM/OM' = OA/OB' = AB/A'B'$ (*Art.* 180).

(182) THEOREM. — *If* ABCD *and* A′B′C′D′ *are two straight lines such that the circles on* AB, CD *as diameters are respectively the inverses of the circles on* A′B′, C′D′ *as diameters, then*

$$\frac{A'B'.C'D'}{AB.CD} = \frac{B'C'.A'D'}{BC.AD} = \frac{C'A'.B'D'}{CA.BD}.$$

Proof.—Let O be the centre of inversion, and P, Q, P′, Q′ the centres of the circles taken in the above order. Since A′B′C′D′ cuts the circles P′ Q′ orthogonally, its inverse is the circle through O orthogonal to the circles P, Q. Let this circle cut the circles P, Q in the inverse points *a*, *b*, *c*, *d*. Then by *Art.* 180, *Cor.* iii,

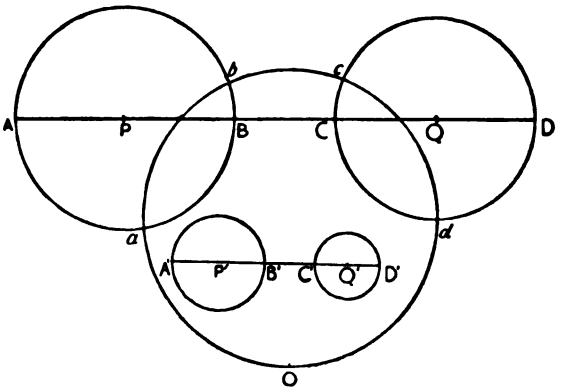
$$\frac{A'B'.C'D'}{ab.cd} = \frac{B'C'.A'D'}{bc.ad} = \frac{C'A'.B'D'}{ca.bd}.$$

Again, if the circles P, Q be regarded as their own inverses, ABCD, which cuts them orthogonally, will have as its inverse the circle which passes through O and cuts them

orthogonally, *i.e.* the circle O*abcd*. Therefore *a*, *b*, *c*, *d* are the inverses of A, B, C, D respectively. Therefore

$$\frac{ab.cd}{AB.CD} = \frac{bc.ad}{BC.AD} = \frac{ca.bd}{CA.BD}.$$

Therefore $$\frac{A'B'.C'D'}{AB.CD} = \frac{B'C'.A'D'}{BC.AD} = \frac{C'A'.B'D'}{CA.BD}.$$



Cor. i.—If TV, T'V' are two direct or two transverse common tangents to the pairs of circles P, Q and P', Q' respectively, $T'V'^2/TV^2$ = each of the above ratios.

Proof.—Let TV, T'V' be direct common tangents, then

$$TV^2 = PQ^2 - (PT - QV)^2$$
$$= (PQ + PT - QV)(PQ - PT + QV) = CA.BD.$$

Similarly $T'V'^2 = C'A'.B'D'$.

Again, let TV, T'V' be transverse common tangents. Then

$$TV^2 = PQ^2 - (PT + QV)^2$$
$$= (PQ - PT - QV)(PQ + PT + QV) = BC.AD.$$

Similarly $T'V'^2 = B'C'.A'D'$.

Cor. ii.—If A, B, C, D are four circles and A', B', C', D' their respective inverses, and if $\overline{AB}$, $\overline{A'B'}$ denote common tangents, either both direct or both transverse, of the pairs

of circles A, B and A′, B′, a similar notation being used in the case of the other pairs, then

$$\frac{\overline{AB}.\overline{CD}}{\overline{A'B'}.\overline{C'D'}} = \frac{\overline{BC}.\overline{AD}}{\overline{B'C'}.\overline{A'D'}} = \frac{\overline{CA}.\overline{BD}}{\overline{C'A'}.\overline{B'D'}}.$$

Proof.—Let $a, b, c, d, a', b', c', d'$ denote the diameters of A, B, C, D, A′, B′, C′, D′ respectively. Then

$$\overline{AB}/\overline{A'B'} = \sqrt{\frac{ab}{a'b'}} \text{ and } \overline{CD}/\overline{C'D'} = \sqrt{\frac{cd}{c'd'}} \text{ (Cor. i).}$$

Therefore

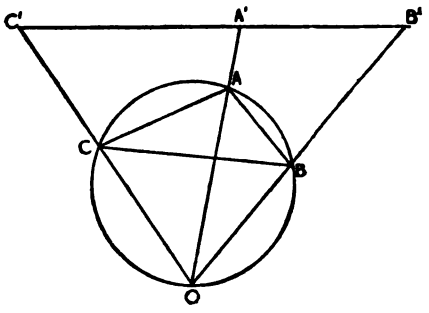
$$\frac{\overline{AB}.\overline{CD}}{\overline{A'B'}.\overline{C'D'}} = \sqrt{\frac{abcd}{a'b'c'd'}}.$$

Similarly it may be proved that

$$\frac{\overline{BC}.\overline{AD}}{\overline{B'C'}.\overline{A'D'}} = \sqrt{\frac{bcad}{b'c'a'd'}}, \text{ and } \frac{\overline{CA}.\overline{BD}}{\overline{C'A'}.\overline{B'D'}} = \sqrt{\frac{cabd}{c'a'b'd'}}.$$

The following articles illustrate the application of the last three theorems.

(183) THEOREM.—*If* α, β, γ *are perpendiculars drawn*



from any point O *on a circle to the sides* BC, CA, AB *respectively of a triangle inscribed in the circle, then*

$$\frac{BC}{\alpha} + \frac{CA}{\beta} + \frac{AB}{\gamma} = 0.$$

Proof.—Let A′, B′, C′ be the inverses of A, B, C respectively with respect to O as centre of inversion. Then since O, A, B, C are concyclic, A′, B′, C′ are collinear (*Art.* 170). Let p be the perpendicular from O on A′B′C′. Then (*Art.* 181)

$$\frac{BC}{\alpha}+\frac{CA}{\beta}+\frac{AB}{\gamma}=\frac{B'C'}{p}+\frac{C'A'}{p}+\frac{A'B'}{p}$$
$$=\frac{B'C'+C'A'+A'B'}{p}=(\text{by } \textit{Art.}\ 7)\ \frac{0}{p}=0.$$

The signs of AB, BC, CA are to be taken as the same as those of the respectively corresponding segments A′B′, B′C′, C′A′. (See *Art.* 12.)

(184) THEOREM (Ptolemy's).—*If* ABCD *is a cylic quadrilateral—*

$$AB.CD+BC.AD+CA.BD=0.$$

Proof.—Let the collinear points A′, B′, C′, D′ be the respective inverses of A, B, C, D with respect to a point on the circumscribing circle as centre of inversion. Then (*Art.* 180, iii)—

$$\frac{AB.CD}{A'B'.C'D'}=\frac{BC.AD}{B'C'.A'D'}=\frac{CA.BD}{C'A'.B'D'}=m.$$

Therefore $AB.CD+BC.AD+CA.BD$
$$=m(A'B'.C'D'+B'C'.A'D'+C'A'.B'D')=0$$
(*Art.* 10).

The sides and diagonals of the quadrilateral are to be regarded as having the same signs as the corresponding segments of the line A′B′C′D′.

(185) THEOREM (Casey's, and an extension of Ptolemy's). —*If four circles, denoted by* A, B, C, D, *touch a fifth circle, then the notation of* Art. 182, Cor. ii, *being adopted—*

$$\overline{AB}.\overline{CD}+\overline{BC}.\overline{AD}+\overline{CA}.\overline{BD}=0.$$

Proof.—Let the circles A′, B′, C′, D′ be the respective inverses of A, B, C, D, with respect to a point on the fifth

circle as centre of inversion. The fifth circle inverts into a straight line touching A′, B′, C′, D′.

Therefore $\overline{A'B'}.\overline{C'D'} + \overline{B'C'}.\overline{A'D'} + \overline{C'A'}.\overline{B'D'} = 0$ (*Art.* 10).

But (*Art.* 182, *Cor.* ii)—

$$\frac{\overline{AB}.\overline{CD}}{\overline{A'B'}.\overline{C'D'}} = \frac{\overline{BC}.\overline{AD}}{\overline{B'C'}.\overline{A'D'}} = \frac{\overline{CA}.\overline{BD}}{\overline{C'A'}.\overline{B'D'}}.$$

Therefore $\overline{AB}.\overline{CD} + \overline{BC}.\overline{AD} + \overline{CA}.\overline{BD} = 0.$

EXERCISES.

CHAPTER I.

(1) If A, B, C are any three points on a straight line, $AB^2 = BC^2 + CA^2 + 2BC.CA$.

(2) If A, B, C are any three points on a straight line, and M the middle point of AB, $MC^2 - MB^2 = AC.BC$.

(3) If A, B, C, D are any four points on a straight line, then—

$$AB^2 - CD^2 = (AC - BD)(AD - BC);$$
$$AC^2 - BD^2 = (AB - CD)(AD - CB);$$
$$AD^2 - BC^2 = (AB - DC)(AC - DB);$$
$$\frac{AB^2 - CD^2}{BC^2 - AD^2} = \frac{BA + CD}{BC + AD};$$
$$\frac{BC^2 - AD^2}{CA^2 - BD^2} = \frac{CB + AD}{CA + BD};$$
$$\frac{CA^2 - BD^2}{AB^2 - CD^2} = \frac{AC + BD}{AB + CD};$$
$$(AB + CD)(BC + AD)(CA + BD) = -(AB - CD)(BC - AD)(CA - BD);$$
$$AB^2 = AC^2 + CD^2 + DB^2 + 2AC.CD + 2CD.DB + 2DB.AC;$$
$$AB.AC + BC.BA + CA.CB + DB.DC + DC.DA + DA.DB = DA^2 + DB^2 + DC^2.$$

(4) If A, B, C, D be any four points on a straight line,

M the middle point of AB and N the middle point of CD, then—

$$2MN.CD = AD.BD - AC.BC;$$
$$NA^2 - NB^2 = AC.AD - BC.BD;$$
$$AB^2 - CD^2 = 4CM.DM - 4AN.BN.$$

(5) Given A, B, C, D four points on a straight line, find a fifth point E on the line such that $AE + BE + CD = 0$.

(6) Show that *Euc.* i. 29 is a particular case of *Euc.* i. 32.

(7) Show that *Euc.* iii. 32 is a particular case of *Euc.* iii. 21.

CHAPTER II.

(1) If ABC is a triangle inscribed in a circle of which BD is a diameter, and if DE be drawn perpendicular to AC, prove $CE^2 - EA^2 - AB^2 + BC^2 = 0$.

Of what theorem is this a particular case?

(2) If L, L′, X; M, M′, Y; N, N′, Z are triads of points on the sides BC, CA, AB respectively of a triangle, and if the perpendiculars to the sides at L, M, N are concurrent, and also the perpendiculars to the sides at L′, M′ N′, and if $LX/XL' = MY/YM' = NZ/ZN'$, then the perpendiculars to the sides at X, Y, Z are also concurrent.

(3) If a line drawn parallel to the side BC of a triangle ABC meets CA in M and AB in N, then BM, CN, and the median AA′ are concurrent.

(4) If ABC is a triangle right-angled at A, then the perpendicular from A on BC, and the connectors of B and C with the corners of the squares described on CA and AB respectively, are concurrent.

(5) Prove Menelaus' theorem by drawing perpendiculars from the vertices of the triangle to the transversal.

(6) Into what proposition of Euclid does Menelaus' theorem resolve itself if the transversal is drawn parallel to one of the sides of the triangle?

(7) In the figure of *Art.* 16, prove—

$$\text{(i)}\ \frac{BL.CN.PM}{LC.NP.MB} = 1;$$

$$\text{(ii)}\ \frac{CM.AN.PN.BM}{MA.NB.NC.MP} = 1.$$

(8) In the figure of *Art.* 17, prove—

$$\text{(i)}\ \frac{CA}{AB} = -\frac{NL.MC}{LM.BN} = \frac{NL.MA.LC}{LM.NA.LB};$$

$$\text{(ii)}\ \frac{AB.MC}{CA.BN} + \frac{BC.MA}{CA.BL} = 1.$$

(9) If a transversal cuts the sides BC, CA, AB of a triangle in L, M, N respectively, and if AB = BC, then LM.AN = MN.CL.

(10) If a transversal cuts the sides BC, CA, AB of an equilateral triangle in L, M, N respectively, then $\frac{BN}{MC} + \frac{AN}{LC} = 1$.

(11) In the figure of *Art.* 16, the intersections of BC and MN, CA and NL, AB and LM are collinear.

(12) State and prove the converse of the theorem in the last exercise.

(13) If L is a fixed point on the side BC of a triangle ABC, and M, N are variable points on CA, AB respectively, such that AL, BM, CN are concurrent, then MN passes through a fixed point on BC.

(14) If L is a fixed point on the side BC of a triangle ABC, and M, N are variable points on CA, AB respectively,

such that L, M, N are collinear, then BM, CN intersect on a fixed straight line through A.

(15) In the figure of *Art.* 16, prove—

(i) $\frac{BL}{LC}=\frac{MA.PB}{CA.MP}=\frac{AB.PN}{AN.CP}$;

(ii) $\frac{AB.BC.NP.PL}{BN.LB.PC.AP}=1$;

(iii) $\frac{BC.LA.PM}{CL.AP.MB}=-1$;

(iv) $\frac{BC.LA.PC.NA}{CL.AP.CN.AB}=1$;

(v) $\frac{BC.PM}{CL.MB}-\frac{CM.PN}{MA.NC}=1$.

(16) In the figure of *Art.* 16, prove—

(i) $\frac{AP.BP.CP}{LP.MP.NP}=\frac{AB.BC.CA}{AN.BL.CM}$;

(ii) $\frac{AP}{PL}=\frac{AN}{NB}+\frac{AM}{MC}$;

(iii) $\frac{CP}{CN}=\frac{PL}{AL}+\frac{PM}{BM}$;

(iv) $\frac{PL}{AL}+\frac{PM}{BM}+\frac{PN}{CN}=1$.

(17) If L, L′; M, M′; N, N′ are pairs of points on the sides BC, CA, AB respectively of a triangle, and if the middle points of the sides are also the middle points of the segments LL′, MM′, NN′ respectively, then—

(i) The perpendiculars to the sides at L′, M′, N′ are concurrent if the perpendiculars at L, M, N are concurrent;

(ii) AL′, BM′, CN′ are concurrent if AL, BM, CN are concurrent;

(iii) L′, M′, N′ are collinear if L, M, N are collinear.

(18) If L, L′; M, M′; N, N′ are pairs of points on the sides BC, CA, AB respectively of a triangle, then if any

three of the following statements are true, the fourth is also true :—

L, M, N are collinear ;
L′, M′, N′ are collinear ;
AL′, BM, CN are concurrent ;
AL, BM′ CN′ are concurrent.

(19) If L, L′; M, M′; N, N′ are pairs of points on the sides BC, CA, AB respectively of a triangle, and if L, L′, M, M′, N, N′ are concyclic, then—

(i) The perpendiculars to the sides at the middle points of LL′, MM′, NN′ are concurrent ;

(ii) The perpendiculars to the sides at L′, M′, N′ are concurrent if the perpendiculars at L, M, N are concurrent ;

(iii) AL′, BM′, CN′ are concurrent if AL, BM, CN are concurrent.

(20) If L, L′ are two points on the side BC of a triangle ABC, such that $BL/LC = -BL'/L'C$, and if the circle through L touching AL′ at A cuts BC, CA, AB in P, Q, R respectively, then AP, BQ, CR are concurrent.

(21) If L, L′; M, M′; N, N′ are pairs of points on the sides BC, CA, AB of a triangle, such that the following are triads of concurrent lines, AL′, BM, CN ; BM′, CN, AL ; CN′, AL, BM, then—

$$\frac{BL'.CM'.AN'}{L'C.M'A.N'B} = 1 \Big/ \left(\frac{BL.CM.AN}{LC.MA.NB}\right)^2.$$

(22) If L, L′; M, M′; N, N′ are pairs of points on the sides BC, CA, AB respectively of a triangle, such that the following are triads of collinear points, L′, M, N; M′, N, L; N′, L, M, then—

$$\frac{BL'.CM'.AN'}{L'C.M'A.N'B} = -1 \Big/ \left(\frac{BL.CM.AN}{LC.MA.NB}\right)^2.$$

(23) If AL_1, BM_1, CN_1; AL_2, BM_2, CN_2; . . . AL_n, BM_n, CN_n are triads of lines drawn from the vertices of

a triangle ABC to the opposite sides, and determining by their intersection triangles t_1, t_2, . . . t_r such that t_2 is circumscribed to t_1, t_3 to t_2, . . . t_r to t_{r-1}, and if $\frac{BL_r.CM_r.AN_r}{L_rC.M_rA.N_rB}$ be denoted by c_r, then

$$c_1 = c_2^{-\frac{1}{2}} = c_3^{\frac{1}{2}} = \ldots = c_r^{(-\frac{1}{2})^{r-1}}.$$

(24) If L_1, M_1, N_1; L_2, M_2, N_2; . . . L_r, M_r, N_r are triads of points lying on the sides BC, CA, AB respectively of a triangle, and determining triangles t_1, t_2 . . . t_r such that t_2 is inscribed in t_1, t_3 in t_2, . . . t_r in t_{r-1}, and if $\frac{BL_r.CM_r.AN_r}{L_rC.M_rA.N_rB}$ be denoted by m_r, then

$$m_1 = -m_2^{-\frac{1}{2}} = m_3^{\frac{1}{2}} = \ldots = (-1)^{r-1} m_r^{(-\frac{1}{2})^{r-1}}.$$

(25) The tangents to the circumcircle at the vertices of a triangle meet the opposite sides in collinear points.

(26) If a transversal cuts the sides AB, BC, CD, DA of a quadrilateral in L, M, N, O respectively, then

$$\frac{AL}{LB}.\frac{BM}{MC}.\frac{CN}{ND}.\frac{DO}{OA} = 1.$$

(27) Extend the theorem of the last exercise to any polygon.

(28) If L, M, N are the feet of perpendiculars drawn from any point on a circle to the sides BC, CA, AB respectively of a triangle inscribed in the circle, then

$$BL^2 - LC^2 + CM^2 - MA^2 + AN^2 - NB^2 = \frac{BL}{LC}.\frac{CM}{MA}.\frac{AN}{NB} + 1.$$

(29) State and prove the converse of *Art.* 19.

(30) Prove *Art.* 19 when PL, PM, PN make any equal angles (measured the same way round) with the sides.

(31) The circumcircles of two triangles ABC, AB'C' which have a common angle at A meet again at P; prove

that the feet of the perpendiculars from P to AB, BC, CA, B′C′ are collinear.

(32) Hence prove that the circumcircles of the four triangles formed by four intersecting straight lines have a common point.

(33) If PQ is a diameter of the circumcircle of the triangle ABC, the pedal lines of the points P, Q with respect to the triangle ABC intersect perpendicularly.

(34) If AP is a diameter of, and Q any point on, the circumcircle of a triangle ABC, the angle PAQ is equal to the angle between BC and the pedal line of Q.

(35) P is a point on the circumcircle of a triangle ABC; the perpendiculars from P on BC, CA, AB meet the circumcircle in Q, R, S respectively; prove that AQ, BR, CS are parallel to the pedal line of P.

(36) D, E are two points on the circumference of a semicircle described on AB as diameter; AE, BD intersect in C, and AD, BE in H; prove that CH is perpendicular to AB.

(37) PQRS is a parallelogram; PT, RT are drawn perpendicular to PQ, QR respectively, and intersect at T; prove that ST is perpendicular to PR.

(38) In the figure of *Art.* 20, $2A'D.BC = AB^2 - CA^2$, where A′ is the mid-point of BC.

(39) ABC is a triangle, H its orthocentre, and AP a diameter of the circumcircle; prove that BHCP is a parallelogram.

(40) If AP, BQ, CR are diameters of the circumcircle of a triangle ABC, then PA′, QB′, RC′ (where A′, B′, C′ are the mid-points of the sides) are concurrent in the orthocentre.

(41) If in the figure of *Art.* 20, the circumcircle of EFA cuts DE again in F′ and DF again in E′, then DF = DF′, and DE = DE′.

(42) In the figure of *Art.* 20—

(i) AB.AF = AD.AH = AC.AE;

(ii) AD.HD = DF.DE = BD.DC;

(iii) $HD^2 = BD.DC - BH.HE$;

(iv) $BC^2 = BH.BE + CH.CF$;

(v) $BC^2 + CA^2 + AB^2 = 2(AD.AH + BE.BH + CF.CH)$.

(43) If in the figure of *Art.* 20 tangents t_1, t_2, t_3 be drawn from A, B, C to the circumcircles of BHC, CHA, AHB respectively, and if the sides of ABC be denoted by a, b, c, then

$$a^2 + b^2 + c^2 = t_1^2 + t_2^2 + t_3^2.$$

(44) ABC is a triangle, H its orthocentre, and A′ the mid-point of BC; HA′ and AD meet the circumcircle in P and Q respectively; prove that PQ is parallel to BC.

(45) Construct a triangle having given the orthocentre, the circumcentre, and a vertex.

(46) Having giving the base and vertical angle of a triangle, find the locus of (i) the orthocentre and (ii) the centroid.

(47) The nine-point circle bisects all straight lines drawn from the orthocentre to the circumference of the circumcircle.

(48) If H is the orthocentre, and P any point on the circumcircle of a triangle ABC, the pedal line of P bisects PH.

(49) The orthocentres of the four triangles formed by four intersecting straight lines are collinear.

(50) The perpendiculars drawn from the mid-points of the sides of a triangle to the opposite sides of the pedal triangle meet in the centre of the nine-point circle.

(51) The perpendiculars drawn from the vertices of a triangle to the opposite sides of the pedal triangle meet in the centre of the circumcircle.

(52) In a triangle ABC, the median BB′ is greater than, equal to, or less than the median CC′ according as CA is less than, equal to, or greater than AB.

(53) If AA′ is a median of the triangle ABC, the angle CAA′ is greater than, equal to, or less than the angle BAA′ according as CA is less than, equal to, or greater than AB.

(54) The sum of the three medians of a triangle is less than the perimeter but greater than three-fourths of the perimeter of the triangle.

(55) If in the figure of *Art.* 20, O, O_1, O_2, O_3 are the circumcentres, and G, G_1, G_2, G_3 the centroids of the triangles ABC, BHC, CHA, AHB respectively, and N is the centre of the nine-point circle of ABC, then—

(i) O_1G_1NA, O_2G_2NB, O_3G_3NC are quartets of collinear points;

(ii) The six connectors of O, O_1, O_2, O_3 are respectively parallel and equal to the six connectors of A, B, C, H;

(iii) Any one of the four points O, O_1, O_2, O_3 is the orthocentre of the triangle formed by joining the other three;

(iv) Any one of the four points G, G_1, G_2, G_3 is the orthocentre of the triangle formed by joining the other three;

(v) H, A, B, C are the circumcentres of the four triangles formed by the six connectors of O, O_1, O_2, O_3;

(vi) The six connectors of O, O_1, O_2, O_3 bisect and are bisected by the six connectors of H, A, B, C;

(vii) The four triangles formed by the connectors of

O, O_1, O_2, O_3, and the four triangles formed by the connectors of H, A, B, C have the same nine-point circle.

(56) In the figure of *Art.* 20—

(i) Triangle ABC : triangle BHC = AB.AC : HB.HC;

(ii) AD.BH.CH = HD.BA.CA;

(iii) BD.CH.CA = CD.BH.BA;

(iv) $abc = a.BH.CH + b.CH.AH + c.AH.BH$;

(v) $4\Delta = AH.BC + BH.CA + CH.AB$;

(vi) $AH^2 + BC^2 = BH^2 + CA^2 = CH^2 + AB^2 = 4R^2$;

(vii) $\frac{AD.BE.CF}{abc} = \frac{\Delta}{2R^2}$.

(57) If two triangles are so placed that the connectors of corresponding vertices are parallel, the intersections of corresponding sides are collinear.

(58) Two triangles whose sides are parallel are in perspective.

(59) If A, B, C, P, Q, R are six points such that the connectors AP, BQ, CR are concurrent, the six points of intersection of the following pairs of connectors AB, PQ; BC, QR; CA, RP; AQ, PB; BR, QC; CP, RA, lie three and three on four straight lines.

(60) If ABC, A'B'C', A''B''C'' are three triangles having their vertices A, A', A'' collinear, and their bases lying in one straight line, and if P, Q, P', Q', P'', Q'' are the points of intersection of the following pairs of sides respectively, viz., AB, A'B'; CA, C'A'; A'B', A''B''; C'A', C''A''; A''B'', AB; C''A'', CA, then the connectors PQ, P'Q', P''Q'' are concurrent.

CHAPTER III.

(The Exercises refer chiefly to the figure of *Art.* 30.)

(1) I′, M′, N′ are collinear.

(2) The diameter of the circle circumscribed to any three of the four points I, I_1, I_2, $I_3 = r_3 + r_2 + r_1 - r$.

(3) The radius of the nine-point circle of ABC = $\frac{1}{8}(r_3 + r_2 + r_1 - r)$.

(4) $rr_1 = (s-b)(s-c)$; $r_1 r_2 = s(s-c)$.

(5) $\Delta = \sqrt{rr_1 r_2 r_3}$.

(6) AI bisects the angle OAH where H is the orthocentre of ABC.

(7) A′X.BC + B′Y.CA + C′Z.AB = 0.

Similar statements may be obtained by substituting X_1, Y_1, Z_1, or X_2, Y_2, Z_2, or X_3, Y_3, Z_3 for X, Y, Z.

(8) The following are triads of concurrent lines—

I_1X_1, I_2Y_2, I_3Z_3; IX, I_3Y_3, I_2Z_2;
I_3X_3, IY, I_1Z_1; I_2X_2, I_1Y_1, IZ.

(9) Parallels through the mid-points of the sides of a triangle to the straight lines connecting the opposite vertices with the incentre or one of the excentres are concurrent.

(10) The centres of the circumcircles of AIB, BIC, CIA, AI_1B, BI_2C, CI_3A lie on the circumcircle of ABC.

(11)
$$\begin{aligned} AI.AI_1 &= AI_2.AI_3 = bc; \\ BI.BI_2 &= BI_1.BI_3 = ac; \\ CI.CI_3 &= CI_1.CI_2 = ab. \end{aligned}$$

(12)
$$R r = \frac{abc}{4s}; \quad R r_1 = \frac{abc}{4(s-a)}.$$

(13) If A is a right angle, $r = s - a$, $r_1 = s$, *and* $2rr_1 = bc = 2BX.XC$.

(14) If ρ, ρ_1, ρ_2, ρ_3, be the radii of the in- and ex-circles of the triangle BHC, where H is the orthocentre of ABC, $r - \rho = r_3 - \rho_3 + r_2 - \rho_2 + r_1 - \rho_1$.

(15) If H is the orthocentre of ABC, $AH + BH + CH = 2(R + r)$.

(16) The following are triads of concurrent straight lines :—

AA′, XI, YZ; BB′, YI, ZX; CC′, ZI, XY, where A′, B′, C′ are the mid-points of BC, CA, AB respectively.

(17) If $A = B = C$, $R = 2r$ and $3R = 2r_1$.

(18) $$\frac{2}{AD} = \frac{1}{r} - \frac{1}{r_1}; \quad \frac{2}{BE} = \frac{1}{r} - \frac{1}{r_2}; \quad \frac{2}{CF} = \frac{1}{r} - \frac{1}{r_3};$$
$$\frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF} = \frac{1}{r};$$
where D, E, F are the feet of the altitudes of ABC.

(19) $AI : AI_1 = IL : LI_1 = r : r_1$.

(20) AX_1, BY_2, CZ_3 cut the incircle in points which are the extremities of diameters through X, Y, Z respectively.

(21) If X′, Y′, Z′ are extremities of diameters of the incircle through X, Y, Z respectively, then AX′, BY′, CZ′ are concurrent.

(22) A′I cuts off from AD a length equal to r, where AD is an altitude of ABC.

(23) Triangle AIX_1 : triangle $AID = r_1 : AD$.

(24) The area of triangle $I_1I_2I_3 = 2Rs$.

(25) The difference in area between triangles UVW and ABC is equal to $NI.s$, where N is the centre of the nine-point circle of ABC.

(26) The circle having I as centre and radius equal to $\sqrt{2}.r$ cuts the sides of ABC and the circles described on AH, BH, CH as diameters in twelve points which connect in six pairs through the orthocentre.

CHAPTER IV.

(1) If in the triangle ABC the antiparallels through A to CA and AB meet BC in P and Q, $AP = AQ$.

(2) If in the triangle ABC, PQ is antiparallel to BC, the median and symmedian through A of the triangle ABC are respectively the symmedian and median through A of the triangle APQ.

(3) The connector of the two points in which the arms of an angle are cut by a circle having its centre on one of two isogonal conjugates with respect to the angle and passing through the vertex of the angle, is perpendicular to the other isogonal conjugate.

(4) The sides of the triangle formed by joining the feet of the perpendiculars from K (the symmedian point) on the sides of the triangle ABC are perpendicular to the medians of ABC.

(5) The sides of the triangle formed by joining the feet of the perpendiculars from G (the centroid) on the sides of the triangle ABC are perpendicular to the symmedians of ABC.

(6) If P, Q are two points on the circumcircle of ABC, such that AP, AQ are isogonal conjugates with respect to the angle BAC, PQ is parallel to BC.

(7) If P, Q are two points on the side BC of a triangle ABC, such that AP, AQ are isogonal conjugates with respect to the angle BAC, then $AP^2 : AQ^2 = BP.PC : BQ.QC$.

(8) If two isogonal conjugates with respect to the angle A of a triangle ABC cut BC in P and Q, and the circumcircle in P′ and Q′, then $CA.AB = AP.AQ' = AQ.AP'$.

(9) If three lines drawn through the vertices of a triangle

meet on the circumcircle, their isogonal conjugates with respect to the angles of the triangle, are parallel.

(10) If PM, PN are perpendiculars from a point P to the arms of an angle BAC, and if Q be the fourth vertex of the completed parallelogram MPN, then AP, AQ are isogonal conjugates with respect to BAC.

(11) If in the figure of *Art.* 45, PM, P'L' meet in X, and PL, P'M' meet in X', then AX, AX' are isogonal conjugates with respect to the angle LAM.

(12) What is the isogonal conjugate of a point on a circle with respect to a given inscribed triangle?

(13) In the figure of *Art.* 46, the sum of the angles BPC, BP'C = 180° + A.

In the following exercises K *is the symmedian point of the triangle* ABC, *and* A', B' C' *are the mid-points of* BC, CA, AB *respectively.*

(14) If from the point where AK meets BC, perpendiculars p, q are drawn to CA, AB respectively, then

$$\frac{p}{b} = \frac{q}{c} = \frac{2\Delta}{b^2 + c^2}.$$

(15) AK : KA'' = $b^2 + c^2 : a^2$, where A'' is the point where AK cuts BC.

(16) If AK, BK, CK meet the opposite sides of ABC in A'', B'', C'' respectively, then

$$\begin{aligned} &BA'' : A''C = c^2 : b^2; \\ &CB'' : B''A = a^2 : c^2; \\ &AC'' : C''B = b^2 : a^2; \\ &BA'' = ac^2/(b^2 + c^2); \\ &A''A' = \frac{a(b^2 - c^2)}{2(b^2 + c^2)}. \end{aligned}$$

(17) If from any point P on AK perpendiculars PM, PN

are drawn to CA, AB, and if Q be the fourth vertex of the completed parallelogram MPN, then PQ is perpendicular to BC.

(18) If from any point P on AK perpendiculars PM, PN are drawn to CA, AB, then M, N are equidistant from the perpendicular drawn from P to BC.

(19) K is the centroid of the triangle formed by joining the feet of the perpendiculars from K on the sides of ABC.

(20) If KL is the perpendicular from K on BC, AA′ cuts it in a point L′, such that L′K = KL.

(21) The connectors of A′, B′, C′ with the mid-points of the altitudes AD, BE, CF respectively, meet in K.

(22) In the figure of *Art.* 53, $KA'' = \frac{a(b^2-c^2)}{a^2+b^2+c^2}$.

(23) In the figure of *Art.* 53, KA″, KB″, KC″ are bisected by AA′, BB′, CC′ respectively.

CHAPTER V.

(1) The circumcentre, the nine-point centre, the orthocentre, and the centroid of a triangle form a harmonic range.

(2) If in the figure of *Art.* 20, FE meets BC in D′, BCDD′ is a harmonic range.

(3) If in the figure of *Art.* 30, ZY meets BC in J, BCXJ is a harmonic range.

(4) In the figure of *Art.* 16 the point of intersection of MN and BC is the harmonic conjugate of L with respect to B and C.

(5) State and prove the converse of the theorem of the last exercise.

(6) If L, M, N are collinear points on the sides BC, CA, AB of a triangle, and L′, M′, N′ their harmonic conjugates with respect to the extremities of these sides, then AL′, BM′, CN′ are concurrent; and conversely.

(7) If LMN is a transversal to the triangle ABC, and L bisects BC, and a line is drawn through A parallel to BC and meeting LMN in P, then PLMN is a harmonic range.

(8) If ABCD is a harmonic range—

(i) $AB^2 = CA.DA + CB.DB$;

(ii) $AB.CD = 2AC.BD = 2AD.CB$;

(iii) $2CA.CB = CD(CA + CB)$.

(9) If ABCD is a harmonic range, O the middle point of AB, and P the point of intersection of the circles described on AB, OD as diameters, PC is perpendicular to AB.

(10) If ABCD is a harmonic range, and O the middle point of AB, $AC.CB = OC.CD$.

(11) Given C, D, P, Q points on a straight line, find A, B points on the line such that ABCD and ABPQ may be harmonic ranges.

(12) If ABCD is a harmonic range, M the middle point of AB, and N the middle point of CD, then

(i) $AM^2 + CN^2 = MN^2$;

(ii) $AC^2 = 2AN.MC$.

(13) If AB is divided harmonically in C, D and also in P, Q, and if O, M, N are the middle points of AB, CD, PQ respectively, then $PC.PD = 2PO.NM$.

(14) Prove that $A.H_2OH$ (notation of Chapters II. and III.) is a harmonic pencil.

(15) Prove that A′.CAKO (notation of Chapters II. and IV.) is a harmonic pencil.

(16) The diagonals of a parallelogram and the straight lines drawn through their point of intersection parallel to the sides of the parallelogram, form a harmonic pencil.

(17) The tangent to the circumcircle at A and the symmedian AK are harmonic conjugates with respect to the sides AB, AC of the triangle ABC.

(18) The straight line which bisects BC of the triangle ABC perpendicularly is divided harmonically by the circumcircle and CA, AB.

(19) In the figure of *Art.* 30, BCLL′, ALII$_1$ and AL′I$_2$I$_3$ are harmonic ranges.

(20) If AL, BM, CN are three concurrent straight lines drawn from the vertices to the opposite sides of a triangle ABC, then if AL is perpendicular to BC, LM and LN are equally inclined to LA.

(21) If ABCD is a harmonic range and P any point on the circle described on AB as diameter, prove that PC : PD is constant.

(22) The first of two circles has its centre on the second; prove that any diameter of the first is divided harmonically by the second and the common chord of the two.

(23) Prove that a tangent to a circle is cut harmonically by the sides of a square described about the circle.

(24) Prove that any tangent to a circle is cut harmonically by its point of contact, a diameter of the circle, and the tangents at the extremities of the diameter.

CHAPTER VI.

(1) What is the polar of the centre of a circle with respect to the circle?

(2) Prove the theorem of *Art.* 69 by drawing PAB a diameter of the circle, and showing that TA, TB are the bisectors of the angle PTT'.

(3) Prove that one, and only one, of the straight lines mentioned in *Art.* 71 cuts the circle.

(4) ABC is a triangle inscribed in a circle; the tangent at A meets BC in P; a second tangent is drawn from P touching the circle at Q and meeting CA, AB in X, Y respectively; prove that PQXY is a harmonic range.

(5) Two tangents are drawn from a point P to a given circle; from any point Q on the chord of contact of these tangents other two tangents QR, QS are drawn to the circle; prove that P, R, S are collinear.

(6) From a point P tangents PQ, PR and a secant PST are drawn to a circle; prove that QR and the tangents at S, T are concurrent.

(7) If P, Q, R are three points such that Q is the pole of RP and R the pole of PQ with respect to a given circle, then P is the pole of QR.

If O is the centre of the given circle, prove that OP, OQ, OR are perpendicular to the sides of the given triangle.

(8) A, B, C, D are four points on a circle; AB, CD meet in E, the tangents at A, B and C, D meet in F and G respectively; FA, FB meet EG in H and I respectively; prove that EGHI is a harmonic range.

(9) Enunciate the reciprocals of the following theorems:—

(i) The theorems of *Arts.* 28, 34, 67, and 69;

(ii) If three points on the sides of a triangle connect concurrently with the opposite vertices, their harmonic conjugates with respect to the extremities of the sides of the triangle are collinear;

(iii) The theorems of Exercises 4, 5, 8.

CHAPTER VII.

(1) How many complete $\left\{\begin{matrix}\text{quadrilaterals}\\ \text{quadrangles}\end{matrix}\right\}$ can be formed having four given $\left\{\begin{matrix}\text{vertices}\\ \text{sides}\end{matrix}\right\}$ in common?

(2) In the figure of *Art.* 80, what is the diagonal triangle of the complete quadrilateral whose sides are LM, MO, ON, NL?

(3) Through a given point draw a straight line which shall be divided harmonically by the point and the sides of a given triangle.

(4) On a given straight line find a point such that the straight lines connecting it with the vertices of a given triangle shall, with the given straight line, form a harmonic pencil.

(5) Prove by applying the theorems of Ceva and Menelaus to the triangle LPQ in the figure of *Art.* 80, that PQST is a harmonic range.

(6) If A, A′ are points on a fixed straight line, and B, B′ points on another fixed straight line, and if AB′, A′B intersect at a fixed point, then the point of intersection of AB, A′B′ lies on a fixed straight line passing through the points of intersection of AA′, BB′.

State and prove the reciprocal theorem.

(7) Prove by means of *Art.* 80 that the diagonals of a parallelogram bisect each other.

(8) Prove the corollary in *Art.* 82 by means of Ceva's theorem.

(9) State the reciprocal of the theorem of *Art.* 82.

(10) The three sides of a triangle self-conjugate with respect to a circle subtend acute angles at the centre of the circle.

CHAPTER VIII.

[*Where an exercise is marked* †, *state and prove the reciprocal theorem, or state and solve the reciprocal problem, as the case may be.*]

(1) If D is the point at infinity on the axis of the range ABCD, then (ABCD) = AC : BC.

(2) If A, B, C, D, E are collinear points, (ABCE)/(ABCD) = (ABDE).

(3) If (ABCD) = (ABEF), then (ABCE) = (ABDF).

(4) If ABCDE and A'B'C'D'E' are two ranges, and if (ABCD) = (A'B'C'D') and (ABCE) = (A'B'C'E'), then (ABDE) = (A'B'D'E').

Write down other existing identities.

(5) If (ABCD) = (A'B'C'D'),

$$\frac{AC.BD}{AD.BC} = \frac{A'C'.B'D'}{A'D'.B'C'},$$

and

$$\frac{AB.CD}{AD.BC} = \frac{A'B'.C'D'}{A'D'.B'C'}.$$

(6) If the sides BC, CA, AB of a triangle are cut by two

transversals in the points L, M, N and L', M', N' respectively, then—

$$(BCLL').(CAMM').(ABNN') = 1.$$

(7) If ABCD . . ., A'B'C'D' . . . are two ranges such that $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \&c.$, then $(ABCD\ldots) = (A'B'C'D'\ldots)$.

(8) If $(ABC\ldots) = (A'B'C'\ldots)$, and if the point at infinity on the first range corresponds to the point at infinity on the second range, then—

$$\frac{AB}{BC} = \frac{A'B'}{B'C'} = \&c.$$

(9) If a segment AA' of given length slides along a given straight line, its extremities A and A' describe two homographic ranges.

(10) If an angle ASA' of given magnitude turns about its vertex S, its arms SA, SA' describe two homographic pencils.

(11) Given three collinear points A, B, C; to find a fourth D, so that (ABCD) may be equal in magnitude and in sign to a given number.

(12) If a transversal is drawn parallel to the ray OD of the pencil O.ABCD, and cutting OA, OB, OC in L, M, N respectively, then $LN : MN = O(ABCD)$.

(13) Given three rays of a pencil, construct a fourth so that the anharmonic ratio of the pencil may be equal to a given number.

(14) Through a given point P draw a straight line cutting the sides of a given triangle in points Q, R, S, such that (PQRS) may be equal to a given number.

(15) If A, A' and B, B' be any two pairs of corresponding points of two projective ranges, and if I be the point at infinity belonging to the first range, and I' the point

corresponding to it in the second range, and if J′ be the point at infinity belonging to the second range, and J its correspondent in the first range, then JA.I′A′ = JB.I′B′.

(16) Prove the theorem of *Art.* 28 by the theorem of *Art.* 105.

†(17) Three fixed straight lines u, u', u'' meet in a point; a variable straight line turning about a fixed point P cuts u and u' in A and A′ respectively; S, S′ are two fixed points on u''; prove that the connectors SA, S′A′ describe two pencils in perspective, and that the locus of their point of intersection is a straight line.

†(18) If the sides of a variable triangle turn respectively about three fixed collinear points, while two of its vertices slide respectively upon two fixed straight lines, the third vertex shall also describe a straight line.

†(19) A simple quadrilateral is divided by a straight line into two others: prove that the intersections of the diagonals of the three lie in a straight line.

†(20) If four tangents to a circle cut a fifth tangent in a harmonic range, they cut every tangent to the circle in a harmonic range.

†(21) If A, B, C, D, A′, B′, C′, D′ are points on a circle such that (ABCD) = (A′B′C′D′), then the intersections of AB′, A′B; AC′, A′C; AD′, A′D are collinear.

(22) State and prove the reciprocal of the theorem of *Art.* 112.

†(23) If PAB, PCD, PEF are three secants to a circle drawn from a point P, then (ABDE) = (ABFC), and (ABCE) = (ABFD).

†(24) If A, B, P, Q, R are five points on a circle such that AB and the tangents at Q, R are concurrent, P.ABQR is a harmonic pencil.

†(25) Given A, B, C, A′, B′, C′ points on a circle, find a point O on the circle such that (OABC) = (OA′B′C′). How many solutions are there?

†(26) Given A, B, C, A′, B′, C′ points on a straight line, find a point O on the line such that (OABC) = (OA′B′C′).

†(27) O, P, Q, R, S are points on a circle; the tangent at O cuts the tangents at P, Q, R, S in A, B, C, D respectively; prove that (ABCD) = O(PQRS).

(28) Employ Menelaus' theorem to prove the theorems of Pascal and Brianchon.

†(29) ABCDEF is a simple hexagon; and P, Q, R, S, T, U, L, M, N are the points of intersection of the pairs of sides AB, CD; BC, DE; CD, EF; DE, FA; EF, AB; FA, BC; AB, DE; BC, EF; CD, FA respectively; prove that if any one of the following thirteen statements is true, the others are true:—

L, M, N are collinear,
(ABPL) = (UBCM),
(BCQM) = (PCDN),
(CDRN) = (QDEL),
(DESL) = (REFM),
(EFTM) = (SFAN),
(FAUN) = (TABL),
D(ABCE) = F(ABCE),
E(BCDF) = A(BCDF),
F(CDEA) = B(CDEA),
A(DEFB) = C(DEFB),
B(EFAC) = D(EFAC),
C(FABD) = E(FABD).

†(30) In the figure of the last exercise, if L, M, N are collinear, then the following are triads of concurrent straight lines:—

PU, AC, LM; QP, BD, MN; RQ, CE, NL; SR, DF, LM; TS, EA, MN; UT, FB, NL.

†(31) In the figure of the last exercise, if L, M, N are collinear, then PS, QT, RU are concurrent.

(32) If A′, B′, C′ are the mid-points of, and D, E, F the feet of the altitudes perpendicular to, the sides BC, CA, AB of a triangle, then the connector of A with the point of intersection of A′F, B′D, the connector of B with the point of intersection of B′D, C′E, and the connector of C with the point of intersection of C′E, A′F are concurrent.

†(33) If ABC, PQR are two triangles circumscribed to the same circle, A(BQCR) = P(BQCR).

(34) By an application of the principle of continuity explained in *Art.* 13, pairs of points or lines being made to coincide, the following theorems may be deduced as particular cases from the theorems of Pascal and Brianchon:—

(i) If A, B, C, D, E are any five { points on / tangents to } a circle, the { points of intersection / connectors } of the pairs AB and DE, BC and EA, CD and the { tangent at / point of contact of } F are { collinear / concurrent }

(ii) If a simple quadrilateral be { inscribed in / circumscribed to } a circle, the { points of intersection / connectors } of the opposite { sides / vertices } and the { point of intersection / connector } of the { tangents at / points of contact of } either pair of opposite { vertices / sides } are { collinear / concurrent }

(iii) The three { points of intersection / connectors } of the { sides / vertices } of a triangle { inscribed in / circumscribed to } a circle, and the { tangents at / points of contact of } the opposite { vertices / sides } are { collinear / concurrent }

(35) Given five { points on / tangents to } a circle, construct by means of the ruler only any number of other { points on / tangents to } the circle.

(36) Given five { points on / tangents to } a circle, construct by means of the ruler only { the tangents at them / their points of contact }

(37) The point of intersection of the diagonals of a simple quadrilateral, which is inscribed in one circle and described about another, has the same polar with respect to both circles.

CHAPTER IX.

(1) A variable circle passing through a fixed point, and having its centre on a given straight line, cuts the line in a system of points in involution.

(2) If AA'BB'CC' are in involution, then—

$$\frac{AB.CA'.AB'.C'A'}{A'B.C'A.A'B'.CA} = 1 \text{ ; and conversely.}$$

(3) If AA'BB'CC' are in involution, then—

$$\frac{AB'.BC'.CA'}{A'B.B'C.C'A} = -1 \text{ ; and conversely.}$$

†(4) If AA'BB'CC' are collinear points, and if ABC'A' and A'B'CA are harmonic ranges, then AA'BB'CC' are in involution.

†(5) If SXAA'BB'CC' are concyclic points, and if S.XA'BC, S.XB'CA, S.XC'AB are harmonic pencils, then the following are pencils in involution :—

S.XABB'CC', S.XBCC'AA', S.XCAA'BB'.

†(6) If S be a point on a circle, and if through any point P inside or outside the circle chords AA', BB', CC'. . . are drawn, then S.AA'BB'CC'. . . is a pencil in involution.

If the polar of P cuts the circle in E, F, then SE, SF are double rays of the involution.

(7) A variable circle having a fixed pole and polar, cuts any straight line through the pole in a system of points in involution.

(8) The tangents drawn to a variable circle having a fixed pole and polar, from any point on the polar, form a pencil in involution.

(9) Angles which have a common vertex and a common fixed bisector form a pencil in involution.

(10) If the double rays of a pencil in involution include a right angle, then any ray and its conjugate make equal angles with either of the double rays.

(11) The six points in which any transversal cuts the three sides of a triangle and any three concurrent straight lines through the vertices, form an involution.

(12) If three straight lines be drawn parallel to each other through the vertices of a triangle, any transversal is cut by them and the sides of the triangle in six points in involution.

(13) The three straight lines drawn through a point

parallel to the sides of a triangle, and the three straight lines joining the point to the vertices of the triangle, form a pencil in involution.

(14) The six straight lines drawn through any point parallel to the six sides of a quadrangle form a pencil in involution.

(15) If six straight lines be drawn parallel to each other through the six vertices of a quadrilateral, any transversal crossing them is cut in six points in involution.

CHAPTER X.

(1) If through C, the mid-point of a segment AB, a perpendicular be drawn to AB, and if through P and Q any two points on the perpendicular, AP, BQ be drawn meeting in D, then the circle described about ABD is orthogonal to any circle passing through P and Q.

(2) Two circles whose centres are A and B are orthogonal if the same straight line is the polar of the point A with respect to circle B, and the polar of the point B with respect to circle A; and conversely.

(3) If A, B, C are the centres of three mutually orthogonal circles, the polar of the point A with respect to the circle B passes through the point C.

(4) If, in the figure of *Art.* 127, RA cuts the circle ABP in S, PS is perpendicular to QR.

(5) The circles having as diameters AH and BC in the figure of *Art.* 20, are orthogonal.

(6) If the common chord of two orthogonal circles is the side of an equilateral triangle inscribed in one of the

circles, it is the side of a regular hexagon inscribed in the other.

(7) If AB is a diameter of a circle, C any point on the diameter, and D and E any points on the circumference, and if circles be described about ACD and BCE intersecting again in F, then the circle described about DEF is orthogonal to the original circle.

(8) Being given the figure of *Art.* 20, draw by means of the ruler only the radical axis of the circum- and nine-point circles of ABC.

(9) Describe a circle to cut two given circles in such a manner that the chords common to the required circle and the given circles may be equal to given straight lines.

(10) The radical axis of two circles bisects the distance between the polars of the centre of each circle with respect to the other circle.

(11) In the figure of *Art.* 30, the perpendicular from A on AI is the radical axis of the incircle I and the excircle I_1; and the perpendicular from A′ on AI_2 is the radical axis of the excircles I_2 and I_3.

(12) If a variable circle touches two fixed circles, its radius bears a constant ratio to the distance of its centre from the radical axis of the fixed circles.

(13) If the circles C and C′, having r and r' respectively as radii, intersect orthogonally at P and Q, then $PQ.CC' = 2rr'$.

(14) If a variable circle whose radius is denoted by r passes through a fixed point P and cuts a fixed circle orthogonally in A, B, and if PM be the perpendicular from P on AB, prove that $r.PM$ varies as AB.

Hence prove that PA.PB varies as AB.

If PA and PB meet the fixed circle in X and Y, prove that PX.PY varies inversely as AB.

(15) The radical centre of the three excircles of a triangle ABC coincides with the incentre of the triangle A'B'C', where A', B', C' are the mid-points of the sides BC, CA, AB respectively.

(16) If two straight lines be divided proportionally, the circles passing through their point of intersection and through pairs of homologous points are coaxal.

(17) Draw the radical axis of a given circle and point.

(18) All circles which pass through a fixed point and cut a given circle orthogonally pass through a second fixed point.

(19) Draw a circle passing through two given points and cutting a given circle orthogonally.

(20) Given two circles of a limiting-point coaxal system, construct another circle belonging to the system.

(21) Draw a circle having a given radius, and cutting two given circles orthogonally.

(22) If A and B are the limiting points of a coaxal system and a tangent AT be drawn from A to one of the circles of the system, the angle ABT is right.

(23) The points of contact of all the tangents drawn from a limiting point to the circles of a coaxal system are collinear.

(24) If A and B are the limiting points of a coaxal system, and r the radius and C the centre of one of the circles, $CA.CB = r^2$.

(25) A limiting point of a system of coaxal circles has the same polar with respect to every circle in the system.

(26) If a given point has the same straight line as polar with respect to every circle of a system, the system is coaxal.

(27) The point of intersection of the diagonals of a simple quadrilateral which is inscribed in one circle and described about another is a limiting point of the coaxal system to which the two circles belong.

(28) Being given a straight line and the two limiting points of a coaxal system, describe two circles belonging to the system and touching the given line.

(29) Circles whose centres are collinear, and which cut a transversal in pairs of conjugate points in involution, are coaxal.

(30) Any straight line drawn through the radical centre of three circles cuts them in six points in involution.

(31) If a chord of one circle touches another circle, the distances of the ends of the chord from the point of contact have to each other the same ratio as the distances of the ends of the chord from either limiting point.

(32) If a transversal be drawn cutting two circles of a limiting-point coaxal system in P, Q, and R, S respectively, then PS and QR subtend equal or supplementary angles at either limiting point.

(33) If A, B are the limiting points and C the centre of one of the circles of a limiting-point system, and if PQ be a chord of circle C passing through A, the points B, P, C, Q are concyclic.

(34) If in the figure of the last exercise, PM, QN, AO are the perpendiculars from P, Q, A respectively on the radical axis of the system, $PM.QN = AO^2$.

(35) If C, C′, C″ be the centres of three coaxal circles,

and if PT, PT′, PT″ be tangents to the circles, C, C′, C″ respectively from any point P, then $C'C''.PT^2 + C''C.PT'^2 + CC'.PT''^2 = 0$.

(36) If a variable chord XY of a circle be divided at P so that PX.PY varies as PA^2, where A is a fixed point, the locus of P is a circle coaxal with the given circle and point.

(37) If a variable chord XY of a given circle subtend a right angle at a fixed point A, the mid-point of XY and the foot of the perpendicular on XY from A describe circles coaxal with A and the given circle.

(38) If a transversal cut one circle in A, B, and another in C, D, and if P, Q, R, S are the points of intersection of the tangents at the pairs of points A and C, B and D, A and D, B and C respectively, then P, Q, R, S lie on a circle coaxal with the two given circles.

(39) State and prove the converse of *Art.* 142.

(40) Apply the converse of *Art.* 142 to prove that the circle to which a triangle is self-conjugate, the nine-point circle, and the circumcircle are coaxal.

(41) If AB, A′B′ are chords of one circle and tangents of another, and if l, m, l' m' are perpendiculars from A, B, A′, B′ respectively to the radical axis of the circles, then $AA' : BB' = \sqrt{l} + \sqrt{l'} : \sqrt{m} + \sqrt{m'}$.

(42) State and prove the converse of the last theorem; and hence prove that if a variable triangle be inscribed in a circle of a coaxal system, and if two of the sides in every position touch two fixed circles of the system, the remaining side touches another fixed circle of the system.

CHAPTER XI.

(1) If two circles whose centres of similitude are E and I, intersect at A, the angle EAI is right.

(2) The straight lines joining a point of intersection of two orthogonal circles with their centres, are inclined at angles of 45° to the straight lines joining the point of intersection with their centres of similitude.

(3) If the polars of one of the centres of similitude of two orthogonal circles intersecting at A, with respect to the circles, meet the line of centres in P and Q, PQ subtends a right angle at A, and AP = AQ.

(4) When two circles are orthogonal, the polar of either centre of similitude with respect to either circle is the polar of the other centre of similitude with respect to the other circle.

(5) If two fixed circles touch at T, and P and Q be variable points one on each circle such that PTQ is always a right angle, then PQ passes through a fixed point.

(6) If Tt is a straight line touching a circle whose centre is M in T and a circle whose centre is m in t, prove that Mt, mT intersect in the middle point of the perpendicular drawn to Tt from the centre of similitude not lying on it.

(7) If two circles whose centres are M and m intersect in P, and if a diameter of circle M be drawn parallel to mP and a diameter of circle m parallel to MP, the connectors of P with the extremities of either diameter pass through the extremities of the other diameter.

(8) The poles of any straight line drawn through a centre of similitude of two circles with respect to the circles are collinear with the centre of similitude.

(9) If Tt is a common tangent of two circles, and if S is the centre of similitude through which it does not pass, the straight line joining the middle point of Tt and the middle point of ST passes through the centre of the circle on which T lies.

(10) If four parallel tangents touch two circles, one circle at P and Q, the other at p and q, the pairs of connectors Pp, Qq and Pq, Qp intersect in the centres of similitude of the circles.

(11) If two parallel chords of two circles have the same ratio as the radii of their respective circles, the straight lines joining their extremities pass through the centres of similitude of the circles.

(12) In the figure of *Art.* 147, the straight lines joining the extremities of the chords drawn through the middle points of RR′, SS′, and rr', ss' intersect in the centres of similitude.

(13) If in the figure of *Art.* 147, S and r' are points of contact of a common tangent, R′S and $r's$ are diameters.

(14) In the figure of *Art.* 147, the connectors RS, R′S′, rs, $r's'$ form a parallelogram one diagonal of which passes through E. The same is true of the connectors RS′, R′S, rs', $r's$.

(15) In the figure of *Art.* 147, MR is perpendicular to the tangent at r.

(16) From a point on the polar of one of the centres of similitude of two orthogonal circles with respect to one of the circles, two tangents are drawn to that circle; prove that the pole of either of these tangents with respect to the other circle lies on the other tangent.

(17) In the figure of *Art.* 147, $Rr.R'r' = Ss.S's' = Tt^2$.

(18) In the figure of *Art.* 147,

$$\frac{\text{ET.E}t}{\text{MT}.mt} = \left(\frac{\text{T}t}{\text{MT} - mt}\right)^2.$$

(19) In the figure of *Art.* 147,

$$\frac{\text{ER.E}r'}{\text{ET}^2} = \frac{mt}{\text{MT}}.$$

(20) S is a centre of similitude of two circles which intersect in L; SL cuts the circles again in M and N; prove that $\text{SL}^2 = \text{SM.SN}$.

(21) State and prove the converses of the theorem and corollaries of *Art.* 150.

(22) If a circle cuts two circles orthogonally, the connector of a point of intersection on one circle with a point of intersection on the other circle passes through a centre of similitude of the two circles.

(23) What does the circle of similitude of two circles become when (i) the two circles are equal, (ii) one of the circles is a point, (iii) the two circles are concentric?

(24) If in the figure of *Art.* 155, R is the radius of circle M and r the radius of circle m, then

$$\frac{\text{IE}}{\text{M}m} = \frac{2\text{R}r}{\text{R}^2 - r^2}.$$

(25) Find the locus of a point which moves so that the ratio of its distances from the centres of two fixed circles is equal to the ratio of the radii of the circles.

(26) PQR, PLM are two variable similar triangles having a common vertex P; the corresponding vertices Q and L are fixed points; the homologous sides QR and LM are fixed in length; find the locus of P.

(27) A common tangent of two circles is divided harmonically by their circle of similitude.

(28) If O is the centre of the circle of similitude in the figure of *Art.* 155, $OM : Om = IM^2 : Im^2 = EM^2 : Em^2$.

(29) The circle described on the third diagonal of a cyclic quadrilateral as diameter is the circle of similitude of the circles described on the other two diagonals as diameters.

(30) The six lines obtained by joining a point on the radical axis of two circles to their centres of similitude, and by drawing from the point four tangents to the circles, form a pencil in involution.

(31) If two circles touch two others, the radical axis of either pair passes through a centre of similitude of the other pair.

(32) The locus of the pole of the chord of contact of the variable circle, with respect to that circle in the figure of *Art.* 157, is the radical axis of the fixed circles.

(33) If in the figure of *Art.* 147, two circles are drawn touching circles M and m at the pairs of points R, r' and R', r, the difference of the radii of the circles so drawn is equal to the sum of the radii of the circles M and m (or in the case where the transversal $rr'RR'$ passes through the internal centre of similitude to the difference of the radii of the circles M and m).

(34) Find a circle which will cut orthogonally the variable circle of *Art.* 157 in all positions.

(35) Find the locus of the point of contact of two variable circles which touch each other and two fixed circles.

CHAPTER XII.

(1) Draw the inverses of the following figures:—

(i) A circle (with respect to itself);

(ii) A triangle and its incircle (with any centre of inversion, and with centre of inversion on the incircle);

(iii) A triangle and its circumcircle (with any centre of inversion, and with centre of inversion on the circumcircle);

(iv) Two intersecting circles and their line of centres (any centre of inversion, point of intersection as centre of inversion);

(v) A circle and two perpendicular diameters (any centre, the centre on the circle);

(vi) Two circles touching (with respect to a circle through the point of contact);

(vii) Two circles touching a straight line and each other at the same point, the smaller circle passing through the centre of the larger (with respect to the larger);

(viii) Two parallel straight lines and a series of circles touching them and each other (any centre, centre on one of the straight lines);

(ix) A triangle and its nine-point circle (any centre) [*vide Art.* 179];

(x) An equilateral triangle (any centre);

(xi) A right-angled triangle (any centre);

(xii) A rectangle (any centre);

(xiii) A square (any centre) [*vide Art.* 179];

(xiv) A regular pentagon (any centre) [*vide Art.* 179].

(2) Invert the following propositions:—

(i) If two angles of a triangle are equal, the sides which are opposite to them are equal [*vide Art.* 179];

(ii) If a transversal crossing two straight lines makes a pair of alternate angles equal, the lines are parallel, and conversely;

(iii) The three interior angles of a triangle are together equal to two right angles;

(iv) If two adjacent sides of a parallelogram are equal, all the sides are equal [*vide Art.* 179];

(v) The opposite angles of a parallelogram are equal;

(vi) The line of centres of two circles in contact passes through the point of contact;

(vii) A straight line perpendicular to a diameter of a circle at one of its extremities touches the circle;

(viii) The angle in a semicircle is a right angle;

(ix) If a chord be drawn in a circle from the point of contact of a tangent, the angles between the chord and tangent are equal to the angles in the alternate segments;

(x) A circle can be described about any rectangle;

(xi) The diagonals of a square bisect each other at right angles;

(xii) The straight lines joining the mid-points of opposite sides of a square bisect each other at right angles;

(xiii) The middle points of the diagonals of a complete quadrilateral are collinear;

(xiv) The bisectors of the angles formed by producing the opposite sides of a cyclic quadrilateral to meet, are at right angles;

(xv) A median of a triangle is divided in a point of trisection by the centroid;

(xvi) The bisectors of the angles of a triangle are concurrent in four points; and if perpendiculars be drawn from these points to the sides, they are equal to each other in four sets of three;

(xvii) The foot of an altitude of a triangle is half-way between the orthocentre and the point in which that altitude produced meets the circumcircle;

(xviii) The circumcentre, centroid, nine-point centre, and orthocentre of a triangle are collinear;

(xix) The theorem of *Art.* 28;

(xx) The theorems of Pascal and Brianchon.

(3) Establish the following propositions by means of the method of inversion:—

(i) Circles drawn through a point orthogonal to a given circle meet in a second point;

(ii) If OP, OQ, OR are diameters of three circles which intersect in O, and again in pairs in the collinear points A, B, C, then O, P, Q, R are concyclic;

(iii) If AB is a diameter of a circle, C any point on the diameter, D, E any points on the circumference, and if circles be described about ACD, BCE intersecting again in F, then the circle described about DEF is orthogonal to the original circle;

(iv) If two circles intersecting at O, P, touch a circle K in Q, R, and if circles be drawn through O, Q and O, R orthogonal to K and intersecting in S, P, Q, R, S are concylic;

(v) If four circles having a common point O touch two circles C, K at four points lying on a circle passing through O, the two touching C intersect with the two touching K at four points which lie on a circle coaxal with C, K;

(vi) If O, A, B, C are four collinear points, and if through O, B and O, C two circles are drawn cutting each other again in D, and if through O, A another circle be drawn cutting the circles OBD, OCD in E, F respectively, then the circles described about BCD, DEF, FCA, ABE have a common point.

(4) The limiting points of a coaxal system are inverse points with respect to every circle of the system.

(5) Any circle drawn through the centre of inversion and a point P which lies outside the circle of inversion cuts the circle of inversion in points which are collinear with the inverse of P.

(6) Show how to invert the sides of a triangle into three equal circles.

(7) When the centre of inversion is the circumcentre of a triangle, the inverses of the vertices of the triangle when joined form a similar triangle.

(8) When the centre of inversion is a point O within a triangle ABC such that $BOC = 60° + A$, $COA = 60° + B$, and $AOB = 60° + C$, the triangle formed by joining the inverses of A, B, C is equilateral.

(9) Determine the nature and position of the inverses of a point, a straight line, and a circle when the circle of inversion is a straight line.

(10) Prove that the inverses of the vertices of a triangle with respect to the circle to which it is self-conjugate are the feet of the altitudes of the triangle.

Hence prove that the circumcircle and the nine-point circle of a triangle are inverses with respect to the circle to which the triangle is self-conjugate.

(11) In the figure of *Art.* 30, the circumcircle of ABC and the nine-point circle of XYZ are inverses with respect to the incircle of ABC.

Also the circumcircle of ABC and the nine-point circle of $X_1Y_1Z_1$ are inverses with respect to the excircle passing through X_1, Y_1, Z_1.

(12) Two intersecting circles have two real circles of antisimilitude which cut the original circles at equal angles and each other orthogonally.

(13) An angle and its bisectors invert into two circles and their circles of antisimilitude.

(14) If one circle lies within another, and if any one complete series of circles can be placed between them so as to touch them and each other, then an infinite number of such series can be so placed.

[Invert the original circles into concentric circles.]

(15) Two circles are inverted into concentric circles; prove that if the original circles are external to each other, the centre of inversion lies inside one and outside the other concentric circle, and that if one of the original circles lies inside the other, the centre of inversion lies inside both or outside both concentric circles.

(16) Show how to invert two circles which are external to each other, so that their inverses may (i) be external to each other, (ii) lie one within the other.

Also show how to invert two circles which lie one within the other, so that their inverses may (i) lie one within the other, (ii) be external to each other.

(17) If a common tangent TT′ be drawn to two inverse circles through the centre of inversion O, and cutting the circle of inversion in P, then $TP^2/T'P^2 = TO/T'O$.

(18) C, C′ are two circles, K, K′ their respective inverses; from the centre of inversion tangents are drawn touching C, K in A, B respectively, and C′, K′ in A′, B′ respectively; from P any point on the circle of inversion tangents are drawn touching C, C′, K, K′ in M, M′, N, N′ respectively; prove that

$$\frac{PM.PM'}{PN.PN'} = \frac{AA'}{BB'}.$$

(19) The points (or circles) A′, B′ and the circle C′ are the inverses, with respect to a certain circle, of the points (or circles) A, B and the circle C respectively; then, if A

and B are inverses with respect to the circle C, A′ and B′ are inverses with respect to the circle C′.

(20) Two circles are cut by their line of centres in the points P, Q and R, S, and by a circle orthogonal to them in the points P′, Q′ and R′, S′ respectively; prove that the connectors PP′, QQ′, RR′, SS′ meet on the orthogonal circle, and that the connectors PQ′, QP′, RS′, SR′ also meet on the orthogonal circle.

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